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A DUAL DIFFERENTIABLE EXACT PENALTY FUNCTION IN FRACTIONAL PROGRAMMING

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A dual differentiable penalty function is introduced for a fractional programming problem. Some results on local and global optimal solution of the problem is obtained via the penalty function method. The result thus generalize those of Han and Mangasarian[1] on non-linear programming.

1. INTRODUCTION

Han and Mangasarian[1] developed a penalty function via the Wolf dual of a given Nonlinear programming problem (NPP) and gave some results on local and global optimum solution to the NPP. In the present note we generalize the results of Mangasarian to incorporate a fractional objective function of the form $f(x)/h(x)$. Observe that in case $h(x) = 1$, then Han and Mangasarian's results can be obtained as particular cases of our results.

2. THE GENERAL FRACTIONAL PROGRAMMING PROBLEM

We consider here the problem,

$$(P) \quad \min_{X \in R^n} \frac{f(x)}{h(x)} \cdot$$

Subject to $g(x) \leqslant 0, \; h(x) \neq 0$

where f and h are functions from n-dimensional Euclidean space R^n into R, and g is from R^n into R^m. The transformation $z_0 = \dfrac{1}{h\ (x)}$ and $z = xz_0$ (Manas[2] and Schaible[3,4]) reduce the problem (P) into the following problem.

$$(P') \quad \min_{(z,z_0) \in R^{n+1}} \quad z_0 f\left(\frac{z}{z_0} \right)$$

subject to

$$g\left(\frac{z}{z_0}\right) \leqslant 0, \left\{ z_0 h\left(\frac{z}{z_,}\right) - 1 \right\} \leqslant 0.$$

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In case f and h are convex functions, the problem (P') is a convex nonlinear programming problem. We consider the Wolfe's dual of problem (P') given as

$$(W) \qquad \max_{(z,z_0,u,v) \in R^{n+2+m}} L(z, z_0, u, v)$$

subject to

$$\nabla_{(z,z_0)} L(z, z_0 u, v) = 0, u \geqslant 0, v > 0$$

where

$$L(z, z_0, u, v) = z_0 f\left(\frac{z}{z_0}\right) + u^T g\left(\frac{z}{z_0}\right) + v^T\left(z_0 h\left(\frac{z}{z_0}\right) - 1\right).$$

As defined in Han and Mangasarian[1] we define a Penalty function

$$(Q) \quad \theta(z, z_0, u, v, \gamma) = L(z, z_0, u, v) - \tfrac{1}{2}\gamma \| \nabla_{(z,z_0)} L(z, z_0, u, v) \|^2$$

and consider the penalty problem :

$$(\theta) \qquad \max_{(z,z_0,u,v) \in R^{n+2+m}} \theta(z, z_0, u, v, \gamma).$$

In case f, h and g are twice differentiable functions, the exact penalty function as introduced above becomes a differentiable function. We have the following result for stationary points of (P), (P'), (W) and (θ). In the following

$$\zeta = (z, z_0) \in R^{n+1}.$$

Theorem 1 (Equivalence of stationary points of (P), (P'), (W) and (θ))—Let f, h and g be twice continuously differentiable at $\bar{x}$.
Then,

(1) If $(\bar{x}, \bar{u})$ is a *K-K-T* point of problem (P) implies that there is a $\bar{v} \in R$ such that $(\bar{z}, \bar{z}_0, \bar{u}, \bar{v})$ is a *K-K-T* point of (P').

(2) If $(\bar{z}, \bar{z}_0, \bar{u}, \bar{v})$ is is a *K-K-T* point of (P') then $(\bar{x}, \bar{u})$ is a *K-K-T* points of (P).

(3) $(\bar{z}, \bar{z}_0, \bar{u}, \bar{v})$ is a stationary point of (W) and $\nabla_{\zeta\zeta} L(z, z_0, u, v)^{-1}$ exists $\Rightarrow (\bar{z}, \bar{z}_0, \bar{u}, \bar{v})$ is a *K-K-T* point of $(P') \Rightarrow (\bar{z}, \bar{z}_0, \bar{u}, \bar{v})$ is a stationary point of (θ) for any γ.

(4) $(\bar{z}, \bar{z}_0, \bar{u}, \bar{v})$ is a stationary point of (θ) for $\gamma \neq 0$ and $1/\gamma$ is not a eigenvalue of $\nabla_{\zeta\zeta} L(\bar{z}, \bar{z}_0, \bar{u}, \bar{v}) \Rightarrow (\bar{z}, \bar{z}_0, \bar{u}, \bar{v})$ is a *K-K-T* point of $(P') \Rightarrow (\bar{z}, \bar{z}_0, \bar{u}, \bar{v})$ is a stationary point of (W).

In the next result we modify the notation as follows : Write $(z, z_0) = X$ and $(u, v) = U$. Then we give the local concavity for $\theta(z, z_0, u, v, \gamma) = \theta(X, U, \gamma)$ in both the variables X and U.

Theorem 2 (Negative semidefiniteness and definiteness of $\theta(\bar{X}, \bar{U}, \gamma)$)—Let $G(X) = \left(g\left(\frac{z}{z_0}\right), z_0 h\left(\frac{z}{z_0}\right) - 1\right)$. Let $(\bar{X}, \bar{U})$ be a *K-K-T* point of (P'). Let f, h and g be continuously differentiable at $\bar{X}$ and let $\nabla_{XX} L(\bar{X}, \bar{U})$ be positive definite with minimum eignevalue $\bar{\rho} > 0$. Then for $\gamma \geqslant \frac{1}{\bar{\rho}}$, $(\bar{X}, \bar{U})$ is a stationary point of (θ) and the Hessian $\nabla^2 \theta(\bar{X}, \bar{U}, \gamma)$ with respect to (X, U) is negative semi-definite. If in addition $\gamma > \frac{1}{\bar{\rho}}$ and $\nabla G(X)$ has linearly independent rows, then $\nabla^2 \theta(\bar{X}, \bar{U}, \gamma)$ is negative definite and hence $(\bar{X}, \bar{U})$ is a strict local maximum of (θ).

Theorem 3 (Stationary point of (θ) as global solutions of (P') and (W))—Let f, h and g be convex and twice continuously differentiable on R^n, let $Y^T \nabla^2 \left(z_0, f\left(\frac{z}{z_0}\right)\right) Y \geqslant \nu \|Y\|^2$ for all $(z, z_0) \in R^{n+1}$ and $Y \in R^{n+1}$ and some $\nu > 0$, and let $\gamma > \frac{1}{\nu}$. For every stationary point $(z(\gamma), z_0(\gamma), u(\gamma), v(\gamma))$ $f(\theta)$, $z(\gamma)$, $z_0(\gamma)$ are independent of γ and $z(\gamma) = \bar{z}$, $z_0(\gamma) = \bar{z}_0$, where $(\bar{z}, \bar{z}_0)$ is the unique solution of (P).

Remarks : In case $h(x) = 1$ in problem (P) we get the Theorems 1, 2 and 4 of Han and Mangasarian[1] as corolaries of our Theorems 1, 2 and 3 respectively. In the pattern of the proof given for the non-linear programming case in Han and Mangasarian[1] we can prove above theorems very easily. So we omit the proof here.

References

1. S. P. Han, and O. L. Mangasarian, *Math. Prog.* **25** (1983), 293-306.
2. M. Manas, *Ekonomicko Mathematicky Obzor.* **4** (1968), 93-99.
3. S. Schaible, *Management Sci.* **22** (1976), 858-67.
4. S. Schaible, *Operations Res.* **24** (1976), 452-61.
5. C. Singh, *J. Optimizational theory Appl.* **33** (2) (1981), 287-94.

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ON A NONLINEAR INTEGRODIFFERENTIAL EQUATION IN BANACH SPACE

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The aim of the present paper is to study the existence, uniqueness and other properties of the solutions of a volterra integrodifferential equation of the form

$$u'(t) + Au(t) = f(t, u(t)) + \int_{t_0}^{t} g\left(t, s, u(s), \int_{t_0}^{s} K[s, \tau, u(\tau)]\, d\tau\right) ds$$
$$u(t_0) = u_0.$$

The theory of the infinitesimal generator of C_0-semigroup in a Banach space, a modified version of contraction mapping principle, integral inequality recently established by Pachpatte are the main tools in our analysis.

1. INTRODUCTION

In this paper we consider the volterra integrodifferential equation of the form

$$u'(t) + Au(t) = f(t, u(t)) + \int_{t_0}^{t} g\left(t, s, u(s), \int_{t_0}^{s} K[s, \tau, u(\tau)]\, d\tau\right) ds, \quad t > t_0 \geqslant 0,$$
$$u(t_0) = u_0 \qquad \ldots(1.1)$$

where $-A$ is the infinitesimal generator of C_0-semigroup $T(t)$, $t \geqslant 0$ on a Banach space B with the norm $\|\cdot\|$, $K \in C[R_+ \times R_+ \times B.B]$, $g \in C[R \times R_+ \times B \times B, B]$ and $f \in C[R_+ \times B, B]$ where $R_+ = [0, \infty]$.

The theory of the existence, uniqueness and other properties of the solutions of various special forms of (1.1) have been extensively studied by using different methods during the past few years[1,4,6,7,10,14,15,16] and the references given therein. The equation (1.1) serves as an abstract formulation of many partial integrodifferential equations which arise in the problems with heat-flow in material with memory, viscoelasticity and many other physical phenomenon[2,4,6,10,14]. The purpose of this paper is to study the existence uniqueness and other properties of the solutions of (1.1). The main tools employed in our analysis are based on the infinitesimal generator of C_0-semigroup, a

modified version of contraction mapping principle and the integral inequality recently established by Pachpatte[13].

2. Preliminaries, Definitions and Statement of Results

Let B be a Banach space with the norm $\|\cdot\|$ and $-A$ is the infinitesimal generator of C_0-semigroup $T(t)$, $t \geqslant 0$, on a Banach space. The set of bounded linear operators $T(t)$, $t \in R_+$, where $R_+ = [0, \infty)$ is a C_0-semigroup on B if

(i) $T(t+s) = T(t)\,T(s) = T(s)\,T(t), \qquad t, s \geqslant 0$

(ii) $T(0) = I$ (the identity operator)

(iii) $T(\cdot)$ is strongly continuous in $t \in R_+$

(iv) $\|T(t)\| \leqslant Me^{wt}$ for some $M > 0$ and real w and $t \in R_+$ (see Martin[9], p. 276).

Before we state our results we give the following definitions used in our subsequent discussion.

A continuous solution u of the integral equation

$$u(t) = T(t-t_0)\,u_0 + \int_{t_0}^{t} T(t-s)\,f(s, u(s))\,ds$$

$$+ \int_{t_0}^{t} T(t-s) \int_{t_0}^{s} g\left(s, \tau, u(\tau), \int_{t_0}^{\tau} K[\tau, \xi, u(\xi)]\,d\xi\right) d\tau\,ds \qquad \ldots(2.1)$$

is called the mild solution of (1.1). The solution $u(t)$ of (1.1) is said to be exponentially asymptotically stable if there exist constants M and $\hat{\alpha}$ such that

$$\|u(t)\| \leqq M\,\|u_0\|\,e^{-\hat{\alpha}(t-t_0)}, \qquad t \geqslant t_0. \qquad \ldots(2.2)$$

holds for $\|u_0\|$ sufficiently small. The solution $u(t)$ of (1.1) is said to be uniformly slowly growing if, for every $\hat{\alpha} > 0$ there exists a constant M, possibly depending on α, such that

$$\|u(t)\| \leqq M\,\|u_0\|\,e^{\hat{\alpha}(t-t_0)}, \qquad t \geqslant t_0. \qquad \ldots(2.3)$$

holds for $\|u_0\| < \infty$.

For convenience, we list the following hypotheses used in our subsequent discussion.

(H_1) For all $t, s \in [t_0, \alpha]$ and $x_i, y_i \in B$, for $i = 1, 2$, there exist nonnegative constants L_i, $i = 1, 2, 3$, such that

$$\|K(t, s, x_1) - K(t, s, x_2)\| \leqslant L_1\,\|x_1 - x_2\|$$

$$\|g(t, s, x_1, y_1) - g(t, s, x_2, y_2)\| \leqslant L_2 [\|x_1 - x_2\| + \|y_1 - y_2\|]$$

$$\|f(t, x_1) - f(t, x_2)\| \leqslant L_3 \| x_1 - x_2\|.$$

(H_2) For every $t' \geqslant 0$ and a constant $c \geqslant 0$, there exist nonnegative constants $L_i(c, t')$ for $i = 1, 2, 3$, such that

$$\|K(t, s, x_1) - K(t, s, x_2)\| \leqslant L_1(c, t')\| x_1 - x_2\|$$

$$\|g(t, s, x_1, y_1) - g(t, s, x_2, y_2)\| \leqslant L_2(c, t') [\|x_1 - x_2\| + \|y_1 - y_2\|],$$

and

$$\|f(t, x_1) - f(t, x_2)\| \leqq L_3(c, t')\|x_1 - x_2\|$$

hold for all $x_i, y_i \in B$, for $i = 1, 2$ with $\|x_i\| \leqq c$, $\|y_i\| \leqq c$ and $t, s \in [0, t']$.

We need the following lemmas in our subsequent discussion.

Lemma 1[11] (p. 28)—Let B be a Banach space. Let D be an operator which maps the elements of B into itself for which D^r is a contraction, where r is a positive integer. Then D has a unique fixed point in B.

Lemma 2[13] (p. 1157-75)—Let $x(t)$, $a(t)$, $b(t)$ and $c(t)$ be real valued nonnegative continuous functions defined on R_+, for which the inequality

$$x(t) \leqslant x_0 + \int_0^t a(s)\, x(s)\, ds$$

$$+ \int_0^t a(s) \left(\int_0^s b(r)\, x(r)\, dr\right) ds$$

$$+ \int_0^t a(s) \left(\int_0^s b(r) \left(\int_0^r c(z)\, x(z)\, dz\right) dr\right) ds$$

holds for all $t \in R_+$, where x_0 is a nonnegative constant, then

$$x(t) \leqslant x_0 \Big[1 + \int_0^t a(s) \exp\left(\int_0^t a(r)\, dr\right)$$

$$\times \Big\{1 + \int_0^s b(r) \exp\left(\int_0^r [b(z) + c(z)] dz\right) dr\Big\} ds\Big].$$

We are now in position to state our main results to be proved in this paper.

Our first theorem deals with the local existence and uniqueness of the solution of (2.1).

Theorem 1—Assume that the hypothesis (H_1) is satisfied. Then for $u_0 \in B$ the initial value problem (1.1) has a unique mild solution $u \in C([t_0, \alpha]; B)$ for $t \geqslant t_0$ such that $t_0 \leqslant t \leqslant \alpha$. Moreover, the mapping $u_0 \to u$ is Lipchitz continuous from B into $C([t_0, \alpha]; B)$.

In the next theorem we give a slightly different version of Theorem 1.

Theorem 2—Let the hypothesis (H_2) be satisfied then for every $u_0 \in B$ there is $t_{\max} \leqslant \infty$ such that the initial value problem (1.1) when $t_0 = 0$ has a unique solution on $[0, t_{\max})$. Moreover, if $t_{\max} < \infty$, then $\lim\limits_{t \uparrow \max} \|u(t)\| = \infty$.

Remark 1 : It is to be noted that Fitzgibbon[4] and Webb[16] have studied the existence, uniqueness and continuation of the solutions of special form of (1.1). Here our method and conditions on the nonlinear functions involved in (1.1) are different from those used in Fitzgibbon[4] and Webb[16]. For our proceeding we shall assume that solutions of (1.1) can be continued on $R_+ = [0, \infty)$.

The following three theorems deals with the boundedness, asymptotic behaviour and growth of the solutions of (1.1).

Theorem 3—Assume that

$$\|T(t - s)\| \leqslant C_1 \qquad \text{...(2.4)}$$

where c is a non-negative constant. Suppose that the functions K, g and f satisfy

$$\|K(t, s, u(s))\| \leqslant M_1(s) \|u(s)\| \qquad \text{.. (2.5)}$$

$$\|g(t, s, u(s), \bar{u}(s))\| \leqq M_2(t) M_3(s) [\|u(s)\| + \|\bar{u}(s)\|] \qquad \text{...(2.6)}$$

$$\|f(t, u(t))\| \leqslant M_2(t) \|u(t)\| \qquad \text{...(2.7)}$$

for all $t, s \in [t_0, \infty)$, $u, \bar{u} \in B$ where $M_1(t)$, $M_2(t)$ and $M_3(t)$ are real valued non-negative continuous functions defined on $[t_0, \infty)$ such that

$$\int_{t_0}^{\infty} M_1(s)\, ds < \infty, \int_{t_0}^{\infty} M_2(s)\, ds < \infty, \int_{t_0}^{\infty} M_3(s)\, ds < \infty. \qquad \text{...(2.8)}$$

Then all solutions of (1.1) are bounded on R_+.

Theorem 4—Assume that

$$\|T(t - s)\| \leqslant C_1 e^{-\hat{\alpha}(t-s)} . \qquad \text{...(2.9)}$$

for a constant $\hat{\alpha} > 0$. Suppose that the functions K, g and f satisfy

$$\|K(t, s, u(s))\| \leqslant M_1(s) e^{-\hat{\alpha}(t-s)} \|u(s)\| \qquad \text{...(2.10)}$$

$$\|g(t, s, u(s), \bar{u}(s))\| \leqslant M_2(t) M_3(s) e^{-\hat{\alpha}(t-s)} [\|u(s)\| + \|\bar{u}(s)\|]. \qquad \text{...(2.11)}$$

$$\|f(t, u(t))\| \leqslant M_2(t)\, \|u(t)\|. \qquad \ldots(2.12)$$

for all $t, s \in [t_0, \infty)$, $\hat{\alpha} > 0$ is a constant, $M_1(t)$, $M_2(t)$ and $M_3(t)$ are as defined in Theorem 3 and satisfy the condition (2.8). Then all the solutions of (1.1) approach to zero as $t \to \infty$.

Theorem 5—Assume that

$$\|T(t - s)\| \leqslant C\, e^{\hat{\alpha}(t-s)} \qquad \ldots(2.13)$$

where $C \geqslant 0$ and $\hat{\alpha} > 0$ are the constants. Suppose that the functions K, g and f satisfy

$$\|K(t, s, u(s))\| \leqslant M_1(s)\, e^{\hat{\alpha}(t-s)}\, \|u(s)\| \qquad \ldots(2.14)$$

$$\|g(t, s, u(s), \bar{u}(s))\| \leqslant M_2(s)\, e^{\hat{\alpha}(t-s)}\, [\|u(s)\| + \|\bar{u}(s)\|]. \qquad \ldots(2.15)$$

$$\|f(t, s, u(t))\| \leqslant M_2(t)\, \|u(t)\| \qquad \ldots(2.16)$$

for all $t, s \in [t_0, \infty)$, $\hat{\alpha} > 0$ is a constant $M_1(t)$, $M_2(t)$ and $M_3(t)$ are defined in Theorem 3 and satisfy the condition (2.8). Then all the solutions of (1.1) are slowly growing.

Remark 2 : It is important to note that in Pachpatte[14] has studied the stability, boundedness and asymptotic behaviour of the solution of (1.1) when $A = A(t)$, $A(t)$ is a one parameter family of closed linear operators. Here our results in Theorem 5 are obtained by using different approach from those in Pachpatte[14].

3. Proofs of Theorems 1 and 2

In the space $C = C([t_0, \alpha], B)$, we define the norm

$$\|u\|_C = \max_{t_0 \leqslant t \leqslant \alpha} \|u(t)\|. \qquad \ldots(3.1)$$

It is easily seen that C with the norm defined in (3.1) is a Banach space, we define a mapping

$$F : C \to C$$

by

$$(Fu)(t) = T(t - t_0)\, u_0 + \int_{t_0}^{t} T(t - s)\, f(s, u(s))\, ds$$

$$+ \int_{t_0}^{t} T(t - s) \int_{t_0}^{s} g\left(s, \tau, u(\tau), \int_{t_0}^{\tau} K[\tau, \xi, u(\xi)]\, d\xi\right) d\tau\, ds$$

$$t_0 \leqslant t \leqslant \alpha. \qquad \ldots(3.2)$$

Clearly, the solution of the equation (1.1) is a fixed point of the operator equation $Fu = u$. Let $u, v \in C$, from (3.2), (3.1) and hypothesis (H_1), we get

$$\|(Fu)(t) - (Fv)(t)\|$$

$$\leqslant \int_{t_0}^{t} ML_3 \|u(s) - v(s)\| \, ds$$

$$+ \int_{t_0}^{t} M \int_{t_0}^{s} L_2 [\|u(\tau) - v(\tau)\| + \int_{t_0}^{\tau} L_1 \|u(\xi) - v(\xi)\| \, d\xi] \, d\tau \, ds$$

$$\leqslant ML_3 \|u - v\|_C (t - t_0) + ML_2 \|u - v\|_C \frac{(t - t_0)^2}{2!}$$

$$+ ML_2 L_1 \|u - v\|_C \frac{(t - t_0)^3}{3!}$$

$$= M(t - t_0)\left[L_3 + L_2 \frac{(t - t_0)}{2!} + L_1 L_2 \frac{(t - t_0)^2}{3!} \right] \|u - v\|_C \qquad \ldots(3.3)$$

where M is a bound of $\|T(t - s)\|$ on $[t_0, \alpha]$. Using (3.2), (H_1) (3.3) and iteration it follows easily that

$$\|(F^n u)(t) - (F^n v)(t)\|$$

$$\leqslant \frac{(t - t_0)^n}{n!} \left\{ M \left[L_3 + L_2 \frac{(t - t_0)}{2!} + L_1 L_2 \frac{(t - t_0)^2}{3!} \right]^n \right\} \|u - v\|_C. \qquad \ldots(3.4)$$

From (3.4) and (3.1), we obtain

$$\|F^n u - F^n v\|_C \leqslant \lambda \|u - v\|_C$$

where

$$\lambda = \frac{1}{n!} \left\{ M\alpha \left[L_3 + L_2 \frac{\alpha}{2!} + L_1 L_2 \frac{\alpha^2}{3!} \right] \right\}^n.$$

For n large enough $\lambda < 1$. Thus, there exists a positive integer n such that F^n is a contraction in C. It follows from Lemma 1 that the operator F has a unique fixed point in C. This fixed point is the desired mild solution of (1.1).

Let v be another mild solution of (1.1) with $v(t_0) = v_0$ on $[t_0, \alpha]$. From (3.2) and hypothesis (H_1), we have

$$\|u(t) - v(t)\|$$

$$\leqslant M \|u_0 - v_0\| + \int_{t_0}^{t} ML \|u(s) - v(s)\| \, ds$$

$$+ \int_{t_0}^{t} \int_{t_0}^{s} ML [\|u(\tau) - v(\tau)\|$$

(equation continued on p. 522)

$$+ \int_{t_0}^{\tau} L \,\|u(\xi) - v(\xi)\|\, d\xi]\, d\tau\, ds$$

where $L = \max\{L_1, L_2, L_3\}$. An application of Lemma 2 yields.

$$\|u(t) - v(t)\|$$

$$\leqslant M \|u_0 - v_0\| [1 + \int_{t_0}^{t} ML \exp(\int_{t_0}^{s} ML\, d\tau)$$

$$\times \{1 + \int_{t_0}^{s} \exp(\int_{t_0}^{T} [1 + L]\, d\xi)\, d\tau\}\, ds]. \qquad \ldots(3.5)$$

From (3.5) and (3.1) it follows,

$$\|u - v\|_C$$

$$\leqslant M \|u_0 - v_0\| [1 + \int_{t_0}^{t} ML \exp(\int_{t_0}^{s} ML\, d\tau)$$

$$\times \{1 + \int_{t_0}^{s} \exp(\int_{t_0}^{\tau} [1 + L]\, d\xi)\, d\tau\}\, ds] \qquad \ldots(3.6)$$

which yields both the uniqueness of u and Lipchitz continuity of the mapping $u_0 \to u$. This completes the proof of Theorem !.

First we shall show that for every $t_0 \geqslant 0$ and $u_0 \in B$, equation (1.1) under the assumptions of theorem has a unique solution u on an interval $[t_0, t_1]$ whose length is bounded below by

$$\delta(t_0, \|u_0\|)$$

$$= \min\left\{1, \frac{\|u_0\|}{[\hat{k}(t_0)\{L_1(\hat{k}(t_0), t_0 + 1) + L_2(\hat{k}(t_0), t_0 + 1)}\right.$$

$$+ L_1(\hat{k}(t_0), t_0 + 1)\, L_2(\hat{k}(t_0), t_0 + 1)\}$$

$$\left. + (N_1(t_0) + N_2(t_0))]\right\} \qquad \ldots(3.7)$$

where $L_i(c, t)$, $i = 1, 2, 3$ are local Lipchitz constants defined as in (H_2), and

$$M(t_0) = \max\{\|T(t)\| : 0 \leqslant t \leqslant t_0 + 1\}$$

$$\hat{k}(t_0) = 2\|u_0\|\, M(t_0)$$

$$N_1(t_0) = \max\{\|f(t, 0)\| : 0 \leqslant t \leqslant t_0 + 1\}$$

$$N_2(t_0) = \max \{\|g(t, s, 0, \int_{t_0}^{s} k[s, \tau, 0 \mid d\tau)\| \mid : 0 \leqslant s \leqslant t \leqslant t_0 + 1\}. \quad \ldots(3.8)$$

Indeed, Let $t_1 = t_0 + \delta(t_0, \|u_0\|)$, where $\delta(t_0, \|u_0\|)$ is defined by (3.7). It is a easy to see that mapping F defined by (3.2) maps the ball of radius $\hat{k}(t_0)$ centred at 0 of $C([t_0, t_1]; B)$ into itself. This is clear from the following estimate. From (3.2), (3.7) and (3.8), we have

$$\begin{aligned}
&\|(Fu)(t)\| \\
&\leqslant M(t_0)\|u_0\| + \int_{t_0}^{t} \|T(t-s)\| \|f(s, u(s)) - f(s, 0)\| ds \\
&\quad + \int_{t_0}^{t} \|T(t-s)\| \int_{t_0}^{s} \|g(s, \tau, u(\tau), \int_{t_0}^{\tau} k[\tau, \xi, u(\xi)] d\xi) \\
&\quad - g(s, \tau, 0, \int_{t_0}^{\tau} k[\tau, \xi, 0] d\xi)\| d\tau\, ds \\
&\quad + \int_{t_0}^{t} \|T(t-s)\| \|f(s, 0)\| ds \\
&\quad + \int_{t_0}^{t} \|T(t-s) \int_{t_0}^{s} \|g(s, \tau, 0, \int_{t_0}^{\tau} k[\tau, \xi, 0] d\xi)\| d\tau\, ds \\
&\leqslant M(t_0)\|u_0\| + M(t_0) L_1(\hat{k}(t_0), t_0 + 1) k(t_0)(t - t_0) \\
&\quad + M(t_0) L_2(\hat{k}(t_0), t_0 + 1) \hat{k}(t_0) \frac{(t - t_0)^2}{2!} \\
&\quad + M(t_0) L_2(\hat{k}(t_0), t_0 + 1) L_1(\hat{k}(t_0), t_0 + 1) \hat{k}(t_0) \frac{(t - t_0)^3}{3!} \\
&\quad + M(t_0) N_1(t_0)(t - t_0) + M(t_0) N_2(t_0) \frac{(t - t_0)^2}{2!} \\
&\leqslant M(t_0)\|u_0\| + M(t_0)(t - t_0)[\hat{k}(t_0)\{L_1(\hat{k}(t_0), t_0 + 1) \\
&\qquad + L_2 \hat{k}(t_0), t_0 + 1) + L_1(\hat{k}(t_0), t_0 + 1) L_2(\hat{k}(t_0), t_0 + 1)\} \\
&\qquad + (N_1(t_0) + N_2(t_0))] \\
&\leqslant M(t_0)\|u_0\| + M(t_0)\|u_0\| \\
&= 2M(t_0)\|u_0\| = \hat{k}(t_0)
\end{aligned}$$

where the last inequality follows from the definition of t_1. The operator F satisfies a uniform Lipchitz condition with the constant

$$L = M(t_0)\,[L_3(\hat{k}(t_0),\ t_0+1) + L_2(\hat{k}(t_0),\ t_0+1)$$
$$+ L_1(\hat{k}(t_0),\ t_0+1)\, L_2(\hat{k}(t_0),\ t_0+1)]$$

and thus by Theorem 1 it has a unique fixed point u in the ball which is the desired solution of (1.1) on $[t_0, t_1]$. From above it is clear that if u is a mild solution of (1.1) when $t_0 = 0$ on the interval $[0, \tau]$ it can be extended to $[0, \tau+\delta]$ with $\delta > 0$ by defining on $[\tau, \tau+\delta]$, $u(t) = w(t)$, where $w(t)$ is a solution of the integral equation

$$w(t) = T(t-\tau)\,u(\tau) + \int_\tau^t T(t-s)\, f(s, w(s))\, ds$$
$$+ \int_\tau^t T(t-s) \int_\tau^s g\left(s, \xi, w(\xi), \int_\tau^\xi k[\xi, \eta, w(\eta) \mid d\eta\right) d\xi\, ds$$
$$\tau \leqslant t \leqslant \tau + \delta. \qquad \text{...(3.9)}$$

Also, δ depends only on $\|u(t)\|$, $\hat{k}(\tau)$, $N_1(\tau)$ and $N_2(\tau)$. Let $[0, t_{max})$ be the maximal interval of existence of the mild solution u of (1.1) when $t_0 = 0$. If $t_{max} < \infty$ then $\lim_{t \to t_{max}} \|u(t)\| = \infty$, since otherwise there is an increasing sequence t_n converges to t_{max} such that $\|u(t_n)\| \leqslant C$ for all n. From the above proof it would imply that for each t_n, near enough to t_{max}, definition of u can be extended from $[0, t_n]$ to $[0, t_n + \delta]$ where $\delta > 0$ is independent of t_n and hence u can be extended beyond t_{max} which contradicts the definition of t_{max}.

For the uniqueness of the local mild solution u of (1.1) when $t_0 = 0$ we suppose that if v is a another mild solution of (1.1) then on every closed interval of $[0, t_{max})$, both u and v exist they coincide by uniqueness argument given in the proof of Theorem 1. Therefore, both u and v have the same t_{max} and on $[0, t_{max})$, $u = v$. This completes the proof of Theorem 2.

4. Proofs of Theorems 3-5

From (2.1), (2.4) – (2.7) we have

$$\|u(t)\| \leqslant \|T(t-t_0)\,u_0\| + \int_{t_0}^t \|T(t-s)\|\, \|f(s, u(s))\|\, ds$$
$$+ \int_{t_0}^t \|T(t-s)\| \int_{t_0}^s \left\|g\left(s, \tau, u(\tau), \int_{t_0}^\tau k[\tau, \xi, u(\xi) \mid d\xi\right)\right\| d\tau\, ds$$
$$\leqslant C_1\|u_0\| + \int_{t_0}^t C_1 M_2(s)\, \|u(s)\|\, ds$$
$$+ \int_{t_0}^t C_1 \int_{t_0}^s M_2(s)\, M_3(\tau)\left[\|u(\tau)\| + \int_{t_0}^\tau M_1(\xi)\, \|u(\xi)\|\, d\xi\right] d\tau\, ds.$$
$$\text{...(4.1)}$$

Using Lemma 2, we get

$$\|u(t)\| \leqslant C_1 \|u_0\| \, [\, 1 + \int_{t_0}^{t} C_1 M_2(s) \exp\,(\int_{t_0}^{s} cM_2(\tau)\, d\tau$$

$$\times \{1 + \int_{t_0}^{s} M_3(\tau) \exp\,(\int_{t_0}^{} [M_3(\xi) + M_1(\xi)]\, d\xi)\, d\tau\}\, ds. \qquad \ldots (4.2)$$

Thus in view of condition (2.8), boundedness of $u(t)$ follows. This completes the proof of Theorem 3.

From (2.1), (2.9) – (2.12), we obtain

$$\|u(t)\| \leqq C_1\, e^{-\hat{\alpha}(t-t_0)} \|u_0\| + \int_{t_0}^{t} C_1\, e^{-\hat{\alpha}(t-s)} M_2(s) \|u(s)\|\, ds$$

$$+ \int_{t_0}^{t} C_1\, e^{-\hat{\alpha}(t-s)} \int_{t_0}^{s} M_2(s)\, M_3(\tau)\, e^{-\hat{\alpha}(s-\tau)}$$

$$[\|u(\tau)\| + \int_{t_0}^{\tau} M_1(\xi)\, e^{-\hat{\alpha}(\tau-\xi)} \|u(\xi)\|\, d\xi]\, d\tau\, ds. \qquad \ldots (4.3)$$

Multiplying both sides of (4.3) by $e^{\hat{\alpha} t}$, we have

$$\|u(t)\|\, e^{\hat{\alpha} t} \leqslant C_1\, e^{\hat{\alpha} t_0} \|u_0\| + \int_{t_0}^{t} C_1\, e^{\hat{\alpha} s} M_2(s) \|u(s)\|\, ds$$

$$+ \int_{t_0}^{t} C_1\, e^{\hat{\alpha} s} \int_{t_0}^{s} M_2(s)\, M_3(\tau) \|u(\tau)\|\, e^{-\hat{\alpha} s} \cdot e^{\hat{\alpha}\tau}\, d\tau\, ds$$

$$+ \int_{t_0}^{t} C_1\, e^{\hat{\alpha} s} \int_{t_0}^{s} M_2(s)\, M_3(\tau)\, e^{-\hat{\alpha} s}\, e^{\hat{\alpha}\tau} \int_{t_0}^{\tau} M_1(\xi)\, e^{-\hat{\alpha}\tau}\, e^{\hat{\alpha}\xi} \|u(\xi)\|\, d\xi\, d\tau\, ds$$

$$= C_1\, e^{\hat{\alpha} t_0} \|u_0\| + \int_{t_0}^{t} C_1\, M_2(s) \|u(s)\|\, e^{\hat{\alpha} s}\, ds$$

$$+ \int_{t_0}^{t} C_1\, M_2(s) \int_{t_0}^{s} M_3(\tau) \|u(\tau)\|\, e^{\hat{\alpha}\tau}\, d\tau\, ds$$

$$+ \int_{t_0}^{t} C_1\, M_2(s) \int_{t_0}^{s} M_3(\tau) \int_{t_0}^{\tau} M_1(\xi) \|u(\xi)\|\, e^{\hat{\alpha}\xi}\, d\xi\, d\tau\, ds. \qquad \ldots (4.4)$$

Applying Lemma 2 to (4.4), we get

$$\|u(t)\|\, e^{\hat{\alpha} t} \leqslant C_1\, e^{\hat{\alpha} t_0} \|u_0\| \, [1 + \int_{t_0}^{t} c\, M_2(s) \exp\,(\int_{t_0}^{s} c\, M_2(\tau)\, d)$$

(equation continued on p. 526)

$$\{1 + \int_{t_0}^{s} M_3(\tau) \exp\left(\int_{t_0}^{\tau} [M_3(\xi) + M_1(\xi)]\, d\xi\right) d\tau\}\, ds]. \quad \ldots(4.5)$$

In view of (2.8), we have

$$\|u(t)\|\, e^{\hat{\alpha} t} \leqslant M_4 \quad \ldots(4.6)$$

where $M_4 > 0$ is a constant. Thus as $t \to \infty$, the solutions of (1.1), approach to zero. This completes the proof of Theorem 4.

From (2.1), (2.13) – (2.16) we obtain,

$$\begin{aligned} \|u(t)\| \leqslant{}& C_1 e^{\hat{\alpha}(t-t_0)} \|u_0\| + \int_{t_0}^{t} C_1 e^{\hat{\alpha}(t-s)} M_2(s) \|u(s)\| u(s)\| \, ds \\ & + \int_{t_0}^{t} C_1 \hat{e}^{\alpha(t-s)} \int_{t_0}^{s} M_2(s) M_3(\tau) e^{\alpha(s-\tau)} \\ & \times [\|u(\tau)\| + \int_{t_0} M_1(\xi) e^{\alpha(\tau-\xi)} \|u(\xi)\|\, d\xi]\, d\tau\, ds. \end{aligned} \quad \ldots(4.7)$$

Multiplying both the sides of (4.7) by $e^{-\hat{\alpha} t}$, we have,

$$\begin{aligned} \|u(t)\|\, e^{-\hat{\alpha} t} \leqslant{}& C_1 e^{-\hat{\alpha} t_0} \|u_0\| + \int_{t_0}^{t} C_1 M_2(s) \| \hat{e}^{-\alpha s}\, ds \\ & + \int_{t_0}^{t} C_1 M_2(s) \int_{t_0}^{s} M_3(\tau) \|u(\tau)\|\, e^{-\hat{\alpha} \tau}\, d\tau\, ds \\ & + \int_{t_0}^{t} C_1 M_2(s) \int_{t_0}^{s} M_3(\tau) \int_{t_0}^{\tau} M_1(\xi) \|u(\xi)\|\, e^{-\hat{\alpha} \xi}\, d\xi\, d\tau\, ds. \end{aligned} \quad \ldots(4.8)$$

An application of Lemma 2, yields

$$\begin{aligned} \|u(t)\|\, e^{-\hat{\alpha} t} \leqslant{}& C_1 e^{-\hat{\alpha} t_0} \|u_0\| [1 + \int_{t_0}^{t} C_1 M_2(s) \exp\left(\int_{t_0}^{s} M_2(\tau)\, d\right) \\ & \{1 + \int_{t_0}^{s} M_3(\tau) \exp\left(\int_{t_0}^{\tau} [M_3(\xi) + M_1(\xi)]\, d\xi\right) d\tau\}\, ds]. \end{aligned} \quad \ldots(4.9)$$

In view of the condition (2.8), we obtain

$$\|u(t)\|\, e^{-\hat{\alpha} t} \leqslant M_5. \quad \ldots(4.10)$$

where $M_5 > 0$ is a constant. The above estimate yields the desired result if we choose M_5 small enough, and the proof of Theorem 5 is complete.

5. EXAMPLE

In this section we give an example to illustrate the applications of our results. Consider the partial integrodifferential equation of the form:

$$u_t(t, x) = u_{xx}(t, x) + \int_0^t \sigma\left(t, s, u_{xx}(s, x), \int_0^s \sigma_1(s, \tau, u_{xx}(\tau, x))\, d\tau\right) ds$$

$$+ \sigma_2(t, u_{xx}(t, x)), \qquad 0 < x < 1,\ t > 0 \qquad \text{..(5.1)}$$

with the given initial and boundary conditions

$$\begin{aligned} u(0, t) &= u(1, 0) \\ u(x, 0) &= u_0(x) \end{aligned} \qquad 0 < x < 1 \qquad \text{...(5.2)}$$

where $\sigma : R_+ \times R_+ \times R \times R \to R$, $\sigma_1 : R_+ \times R_+ \times R \to R$ and $\sigma_2 : R_+ \times R \to R$ are continuous, continuously differentiable with respect to the first argument, uniformly Lipchitz continuous in s, τ and $t \in [0, \infty)$ respectively and satisfy the following conditions;

(a_1) $| \sigma_1(t, s, z_1) - \sigma_1(t, s, z_r) | \leqslant L_1 | z_1 - z_2 |$

(a_2) $| \sigma(t, s, z_1, z_2) - \sigma(t, s, \bar{z}_1, \bar{z}_2) | \leqslant L_2 [| z_1 - \bar{z}_1 | + | z_2 - \bar{z}_2 |]$

(a_3) $| \sigma_2(t, z_1) - \sigma_2(t, z_2) | \leqslant L_3 | z_1 - z_2 |$

for $z_1, z_2, \bar{z}_1, \bar{z}_2 \in R$ and constants $L_1 \geqslant 0$, $i = 1, 2,$.

We first reduce the problem (5.1)—(5.2) to the form (1.1) by making suitable choices of A, f, K and g and illustrate the hypotheses of main results established in section 3.

For, Let $B = L^2(0, 1, R)$; $A_z = z''$;

$$D(A) = \{z \in B,\ z'' \in B;\ z(0) = z(1) = 0\}$$

and $K : R_+ \times R_+ \times B \to B$; $g : R_+ \times R_+ \times B \times B \to B$, $f : R_+ \times B \to B$ are defined as follows

$$f(t, z)(x) = \sigma_2(t, z''(x)), \quad (t, z) \in R_+ \times B$$

$$K(t, s, z)(x) = \sigma_1(t, s, z''(x))$$

$$g(t, s, z, \xi)(x) = \sigma(t, s, z''(x), \xi(x)), \quad 0 < x < 1. \qquad \text{...(5.3)}$$

Equation (1.1) then becomes the abstract formulation of (5.1)-(5.2). Now from (a_1)-(a_3) and (5.3) all the assumptions of Theorem 1 are satisfied and there exists a unique solution of (5.1)-(5.2).

Suppose that the functions involved in (5.3) satisfy the following conditions:

(b_1) $| \sigma_1(t, s, z_1) | \leqslant M_1(s) | z_1 |$

$$(b_2) \quad | \sigma(t, s, z_1, z_2) | \leqslant M_2(t) M_3(s) [| z_1 | + | z_2 |]$$

$$(b_3) \quad | \sigma_2(t, z_1) | \leqslant M_2(t) | z_1 |$$

and also

$$(b_4) \quad | T(t - s) | \leqslant C_1$$

where $M_1(t)$, $i = 1, 2, 3$ are non-negative real-valued continuous functions and satisfying condition (2.8) and $C_1 > 0$, is a constant. Then by Theorem 3 all the solutions of (5.1)–(5.2) are bounded.

Further if the functions in (5.3) satisfy the conditions;

$$(c_1) \quad | \sigma_1(t, s, z_1) | \leqslant M_1(s) e^{-\hat{\alpha}(t-s)} | z_1 |$$

$$(c_2) \quad | \sigma(t, s, z_1, z_2) | \leqslant M_2(t) M_3(s) e^{\hat{\alpha}(t-s)} [| z_1 | + | z_2] |$$

$$(c_3) \quad | \sigma_2(t, z_1) | \leqslant M_2(t) | z_1 |$$

and also

$$(c_4) \quad | T(t - s) | \leqslant C_1 e^{-\hat{\alpha}(t-s)}$$

where function $M_1(t)$, $i = 1, 2, 3$ are same as defined above. Then by Theorem 4 all the solutions (5.1)–(5.2), approach to zero as $t \to \infty$.

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References

1. V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach spaces.* Noordhoff International, Leyden, the Netherlands, 1976.
2. A. Beelent Morante, *Bull. Mat Ital.* 15 (1978), 470–82.
3. C. M. Dafermos and J. A. Nohel, *Comm. in P.D.E.* 4 (1979), 219–78.
4. W. E. Fitzgibbon, *Nonlinear Analysis TMA* 4 (1980), 745–60.
5. A. Friedman, *Partial Differential Equations Holt.* Rinehart and Wiston Inc., New York, 1969.
6. M. L. Heard, *J. Math. Anal. Appl.* 80 (1981), 175–202.
7. T. Kate, *Perturbation Theory Linear Operators.* Springer Verlag, New York, 1966.
8. R. C. MaCasmy, *Q. Appl. Math.* 35 (1977), 163–94.
9. R. H. Martin (Jr.), *Nonlinear Operators and Differential in Banach Space.* Wiley, New York, 1976.
10. R. K. Miller, *Space. Funck. Ekvac.* 18 (1975), 163–94.
11. M. G. Murge, and B.G. Pachpatte, *Analele Stiintifice ale Universitatii Tomul XXX S.I. a Mathematica* 1984, 25–34.
12. J. A. Nohel, Nonlinear Volterra integral equations for heat flow in materials with memory. *MRC. Tech. Summ Report 28), Math. Resecent*, University Wisconsin.

13. B. G. Pachatte, *Indian J. Pure. Appl. Math.* **8** (1977), 1157–75.
14. B. G. Pachpatte, *Bull. Austrial Math. Soc.* **12** (1975), 337–50.
15. A. Pazy, *Semigroups of Linear Operators and Applications to the Partial Differential Equations* (183-205), 1983, Springer Verlag, New York, 1983, pp 183–205.
16. G. F. Webt, *Proc. Am. Math. Soc.* **69** (1978), 255–60.

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EXTENDING THE THEORY OF LINEARIZATION OF A QUADRATIC TRANSFORMATION IN GENETIC ALGEBRA

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In this paper it is shown that the theory of linearization of a quadratic transformation in a genetic algebra A_n, initiated by Holgate[6] and developed by Abraham[1,2], can be extended to include some useful special cases. A general method of constructing basis monomials of the induced linear space B_n is described. Finally proceeding parallel to the original theory, the maximum possible dim B_n as the number of solutions of a diophantine inequality and equality, are obtained.

1. INTRODUCTION

The works of Etherington[4], Schafer[7] and Gonshor[5] led to the introduction of a genetic algebra (GA) in population genetics. A genetic algebra is the algebra A_n which admits a (canonical) basis $\{c_0, c_1, c_2, \ldots, c_n\}$ obeying

$$c_i \, c_j = \sum_{k=0}^{n} \lambda_{ijk} \, c_k, \; i, j = 0, 1, 2, \ldots, n$$

where λ_{ijk} are field elements with the properties

G (i) $\lambda_{000} = 1$

G (ii) $\lambda_{0jk} = 0$, for $k < j$

G (iii) $\lambda_{ijk} = 0$, for $k \leqslant \max(i, j)$, if $i, j > 0$.

In a genetic algebra the transformation $\phi : x \to x^2$ is basic representing the transition from a distribution x of genotypes to the filial distribution x^2 under random mating. Holgate[6] made an important contribution by showing that ϕ can be linearised i.e. the underlying vector space A of a genetic algebra with a canonical basis and with an idempotent can be embedded into a higher dimensional space B (of minimum dimension) such that ϕ has the image $\bar{\phi}$ (a linear transformation) in B. The theory of linearization of a quadratic transformation in a genetic algebra was brought to a more satisfactory form by Abraham[1,2], Abraham[1] has

(a) proved that dim B is independent of a basis of the genetic algebra A,

(b) described a method to generate the monomials required to form a basis of B,

(c) obtained dim B, in terms of dim A exactly, recursively and asymptotically.

He has obtained, as remarked by him, the maximum value of dim B for a genetic algebra of order $(n + 1)$ by assuming that none of the multiplication constants of G.A. is zero. Dim B is too great even for GA's of smaller dimensions, e.g. for $n = 7 + 1 = 8$, dim $B = 27337$, vide Abraham[1]. However, in many concrete genetic situations, the multiplication constants vanish in a definite and simple pattern. In these cases the upper limit on dim B is greatly reduced which cannot be deduced from Abraham's results. There are genetic algebras A_n satisfying

$$c_0^2 = c_0 \qquad \ldots(1_a)$$

and

$$\lambda_{ijk} = 0 \text{ if } 0 \leqslant i, j < k;\ k = 1, 2, \ldots, r \qquad \ldots(1_b)$$

$$\text{(or more generally } \lambda_{ijk} = 0, \text{ if } 0 \leqslant i, j < k \text{ and } k = t_1, t_2, \ldots, t_r) \qquad \ldots(1'_b)$$

$(1'_b)$ expresses the fact that $c_i\, c_j$ has no component with respect to c_k if $\{i, j\} \neq \{0, k\}$, $k = t_1, t_2 \ldots, t_r$.

For infinite random mating populations of diploids with Mendelian segregation such algebras are

(i) gametic and zygotic algebras for a single locus,

(ii) gametic algebras with uncoupled or strictly coupled loci,

(iii) gametic algebras with coupled loci and the linkage distribution $p\,(U)$ satisfying the condition $\Sigma\, p\,(U) < 1$.

For the description of (i), (ii) and (iii) we refer to Wörz-Busekros[8]. It is interesting to note that if (1_b) [or $(1'_b)$] is supplemented by the condition $\lambda_{0kk} = 1/2$, $k = 1, 2, \ldots, r$ (or $k = t_1, t_2, \ldots, t_r$), which is usually the case when we assume Mendelian segregation, then A_n has an r-parametric family of idempotents [vide Theorems 4.6 and 4.8 in Wörz-Busekros[8]]. In a genetic algebra idempotents represent equilibria of a population.

The purpose of this paper is to extend Abraham's theory to obtain a (minimal) linearization even when some multiplication constants of GA are zero. In Section 2.2 and section 3 we use our method to obtain a basis and the dimension of the induced linear space B with respect to a quadratic transformation in A_n satisfying (1_a) and (1_b). We have stated a theorem describing two methods of generating basis monomials necessary to linearize a quadratic transformation in A_n. This theorem is a generalization of Theorem 2 in Abraham[1]. However, we have followed a different line of reasoning for the proof. Finally we have obtained the upper limit on dim B for genetic algebras A_n of dimension $n + 1$ as the number of solutions of a diophantine inequality and equality.

2. Linearization of a Quadratic Transformation

2.1. *Linearizing a Quadratic Transformation in a Baric Algebra*

It is known that the class of baric algebras is wider than that of genetic algebras [vide Wörz-Busekros[8]]. Hence we describe first the linearization of map $\phi : x \to x^2$ for a baric algebra A_n.

Let $x = (1, x_1, x_2, \ldots, x_n)$ be an element of unit weight in A_n. ϕ maps $(1, x_1, x_2, \ldots, x_n)$ into $(1\,\phi, x_1\,\phi, x_2\,\phi, \ldots, x_n\phi)$ where $1\phi = 1$ and $x_k\phi$ are linear combinations of monomials in $x_1, x_2, \ldots, x_n$ of degree $\leqslant 2$. Let $\bar{\phi}$ be the linear mapping extended to the linear space spanned by the monomials $x_1^{i_1}\; x_2^{i_2} \ldots x_n^{i_n}$ such that

$$x_k\, \bar{\phi} = x_k \phi$$

and

$$(x_1^{i_1}\; x_2^{i_2} \ldots x_n^{i_n})\, \bar{\phi} = (x_1\; \bar{\phi})^{i_1} (x_2\; \bar{\phi})^{i_2} \ldots (x\; \bar{\phi})^{i_n}.$$

If there exists a least integer $s \geqslant 1$ for which the set $\{x_k\bar{\phi}, x_k\,\bar{\phi}^2, \ldots, x_k\,\bar{\phi}^s\}$ is linearly dependent, then it spans a subspace L_{x_k}. Let us call it a cyclic subspace (of dim $s-1$) generated by x_k with respect to ϕ. If the finite dimensional cyclic spaces L_{x_k} exist for $k = 1, 2, \ldots, n$; ϕ is said to be linearisable. $L_{x_1} + L_{x_2} + \ldots + L_{x_n} = B_n$ is called the induced linear space. The restriction of $\bar{\phi}$ to B_n is the representation of ϕ. The set of all basis monomials of L_{x_k}, $k = 1, 2, \ldots, n$; form a basis of B_n.

2.2. *Linearization of a Quadratic Transformation in a Genetic Algebra*

Let A_n be a genetic algebra over the real or complex field with a canonical basis $\{c_0, c_1, c_2, \ldots, c_n\}$ satisfying the conditions (1_a) and (1_b).

If $x = c_0 + x_1\, c_1 + \ldots + x_n\, c_n$ is any element (of unit weight) of A_n, then $x^2 = c_0 + x_1^{[2]}$ $c_1 + \ldots + x_n^{[2]}\, c_n$

where

$$x_k^{[2]} = 2\lambda_k\, x_k;\ \lambda_k = \lambda_{00k};\ k = 1, 2, \ldots, r.$$

and

$$x_k^{[2]} = 2x_k\, \lambda_k + \sum_{i,j=0}^{k-1} x_i\, x_j\, \lambda_{ijk}. \text{ where } x_0 = 1,\ k = (r+1), (r+2), \ldots, n.$$

$$\lambda_k = \lambda_{00k}$$

(*equation continued on p. 533*)

$$= 2x_k \lambda_k + 2 \sum_{i=1}^{k-1} \lambda_{i0k} x_i + \sum_{i,j=1}^{k-1} \lambda_{ijk} x_i x_j.$$

Let us introduce

l_r = a linear combination of $x_1, x_2, \ldots, x_r$

l_r^2 = a linear combination of all monomials $x_i x_j$; $i, j = 1, 2, \ldots, r$

l_r^k = a linear combination of all monomials in $x_1, x_2, \ldots, x_r$ of degree k.

We assume that the coefficients in l_r^k are immaterial. Thus they obey the ordinary law of indices e.g. $l_r^m \, l_r^n = l_r^{m+n}$, $(l_r^m)^n = l_r^{mn}$ and also $a\, l_r = l_r$, a being any scalar. In order to get maximum possible dim B, we further assume that the coefficients of every l_r^k are non-zero.

Then

$$\left.\begin{aligned} x_k^{[2]} &= 2\lambda_k x_k, \text{ when } k = 1, 2, \ldots, r \\ &= l_{k-1}^2 + l_k, \text{ when } r < k \leqslant n. \end{aligned}\right\} \qquad \ldots(2)$$

Now $\phi : x \to x^2$ maps $(1, x_1, x_2, \ldots, x_n)$ into $(1, x_1^{[2]}, x_2^{[2]}, \ldots, x_n^{[2]})$, where $x_k^{[2]}$ are given by (2).

Thus

$$\left.\begin{aligned} 1\phi &= 1 \\ x_k \phi &= 2\lambda_k x_k, \; k = 1, 2, \ldots, r \\ x_{r+1}\phi &= l_r^2 + l_{r+1} \\ x_{r+2}\phi &= l_{r+1}^2 + l_{r+2} \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n\phi &= l_{n-1}^2 + l_n. \end{aligned}\right\} \qquad \ldots(3)$$

Let $\bar{\phi}$ be a linear mapping extended to the linear space spanned by the monomials $x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}$ (where i's are non-negative integers and $x_i^0 = 1$), such that $x_k \bar{\phi} = x_k \phi$ and

$$(x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n})\, \bar{\phi} = (x_1 \bar{\phi})^{i_1} (x_2\phi)^{i_2} \ldots (x_2 \bar{\phi})^{i_n}.$$

Obviously the spaces L_{x_k} are one-dimensional cyclic spaces for $k = 1, 2, \ldots, r$. The following theorem gives the linearization of ϕ.

Theorem 1—For $k = r + 1, r + 2, \ldots, n$, $x_k \bar{\phi}^{k-r}$ generates the basis monomials of the cyclic spaces L_{x_k} which form the ascending chain

$$L_{x_r} \subset L_{x_{r+1}} \subset \ldots \subset L_{x_n}.$$

PROOF : By (3) for $k = 1, 2, \ldots, r$

$$x_k \phi = 2 \lambda_k x_k$$

$$\Rightarrow x_k \bar{\phi} = 2 \lambda_k x_k$$

$$\Rightarrow l_r \bar{\phi} = l_r. \qquad \ldots(\text{i})$$

Also by (3), for $k > r$,

$$l_{r+1} \bar{\phi} = l_r^2 + l_{r+1} = x_{r+1} \bar{\phi} \qquad \ldots(\text{ii})$$

$$l_{r+2} \bar{\phi} = l_{r+1}^2 + l_{r+2} = x_{r+2} \bar{\phi} \qquad \ldots(\text{iii})$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$l_n \bar{\phi} = l_{n-1}^2 + l_n = x_n \bar{\phi}. \quad \ldots(n - r + 1).$$

Also

$$l_k^2 \bar{\phi} = (l_k \bar{\phi})^2.$$

Hence

$$x_{r+1} \bar{\phi} = l_r^2 + l_{r+1} = L_{r+1} \quad (\text{say})$$

$$x_{r+2} \bar{\phi}^2 = (l_{r+1} \bar{\phi})^2 + l_{r+2} \bar{\phi}$$

$$= (l_r^2 + l_{r+1})^2 + l_{r+1}^2 + l_{r+2}, \text{ by (ii) and (iii)}$$

$$= (l_r^2 + l_{r+1})^2 + l_{r+2}$$

$$= L_{r+1}^2 + l_{r+2} = L_{r+2} \quad (\text{say})$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

(*equation continued on p.* 535)

$$x_{r+s}\,\overline{\phi}^{s} = L^2_{r+s-1} + l_{r+s} = L_{r+s} \quad \text{(say)}$$

$$\cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots$$

$$x_n\,\overline{\phi}^{n-r} = L^2_{n-1} + l_n = L_n \text{ (say).}$$

Also it is easy to verify that $\overline{\phi}^{s}$ "saturates" x_{r+s} in the sense that $x_{r+s}\,\overline{\phi}^{s+1} = L_{r+s}\,\overline{\phi} = L_{r+s}$ for $s = 1, 2, \ldots, n-r$. L_{r+s} contains all the monomials which form a basis of the cyclic space $L_{x_{r+s}}$. Hence we say that $x_{r+s}\,\overline{\phi}^{s}$ generates a basis of the cyclic space $L_{x_{r+s}}$, $s = 1, 2, \ldots, n-r$. Clearly L_{r+s+1} contains all the monomials of L_{r+s} i.e. $L_{x_{r+s}}$ is a subspace of $L_{x_{r+s+1}}$, for $s = 1,\ldots,2, n-r-1$, which is expressed as n

$$L_{x_{r+1}} \subset L_{x_{r+2}} \subset \ldots \subset L_{x_n}.$$

Consequences of the theorem :

(a) The induced linear space corresponding to $(r+s+1)$ – dim genetic algebra A_{r+s} is $B_{r+s} = L_{x_{r+s}}$, $s = 1, 2, \ldots, n-r$.

(b) If $s_r = x_1 + x_2 + \ldots + x_r$, the basis monomials of $B_{r+1}, B_{r+2}, \ldots, B_n$ (except 1) are generated successively by

$$\left.\begin{aligned}
P_{r+1}(x_1, x_2, \ldots, x_{r+1}) &= s_r^2 + s_{r+1} = P_{r+1} \quad \text{(say)}\\
P_{r+2}(x_1, x_2, \ldots, x_{r+2}) &= P^2_{r+1} + s_{r+2} = P_{r+2} \quad \text{(say)}\\
\cdots\ \cdots\ \cdots\ &\cdots\ \cdots\ \cdots\ \cdots\\
P_n(x_1, x_2, \ldots, x_n) &= P^2_{n-1} + s_n = P_n \quad \text{(say)}
\end{aligned}\right\} \qquad \ldots(4)$$

or equivalently the monomials of $B_{r+1}, B_{r+2}, \ldots, B_n$ (including unity) are generated successively by

$$\left.\begin{aligned}
P_{r+1}(1, x_1, x_2, \ldots, x_{r+1}) &= (1 + s_r)^2 + x_{r+1} = P_{r+1} \quad \text{(say)}\\
P_{r+2}(1, x_1, x_2, \ldots, x_{r+2}) &= P^2_{r+1} + x_{r+2} = P_{r+2} \quad \text{(say)}\\
\cdots\ \cdots\ \cdots\ &\cdots\ \cdots\ \cdots\ \cdots\\
P_n(1, x_1, x_2, \ldots, x_n) &= P^2_{n-1} + x_n \quad \text{(say)}
\end{aligned}\right\} \qquad (5)$$

when $r = 1$, this mode of generating monomials reduces to one given by Abraham[1].

3. The Dimension of Induced Linear Space B_n

In Propositions 2 (a) and 2 (b) below we show that dim B_n can be expressed as the number of solutions of a linear diophantine inequality which can be converted into an equality.

Propositions 2a—Dim B_n is the number of monomials $x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$ such that $a_1, a_2, \ldots, a_n$ are integral non-negative solutions of the inequality

$$a_1 + a_2 + \ldots + a_r + 2a_{r+1} + 2^2 a_{r+2} + \ldots + 2^{n-r} a_n \leqslant 2^{n-r}, \forall n \leqslant r \qquad \ldots(6)$$

2 (b) Dim B_n is the number of integral non-negative solutions of the linear diophantine equation :

$$a_1 + a_2 + \ldots + a_{r+1} + 2a_{r+2} + \ldots + 2^{n-r} a_{n+1} = 2^{n-r}, \forall n \geqslant (r+1). \qquad \ldots(7)$$

PROOF OF PROPOSITION 2 (a) : We define the weight $w(x_i)$ of x_i, $i = 1, 2, \ldots, n$; as

$$w(x_i) = 1, \text{ for } i = 1, 2, \ldots, r$$
$$= 2^{i-r}, \text{ for } i = (r+1), (r+2), \ldots, n$$

and

$$w(x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}) = \sum_{i=1}^{n} a_i w(x_i)$$

when $n = r$, (6) is $a_1 + a_2 + \ldots + a_r \leqslant 1$ which gives $(r+1)$ solutions equal to the number of monomials $1, x_1, x_2, \ldots, x_r$ which span B_r. Similarly we can show that the proposition holds for $n = (r+1)$. Next let us assume that Proposition 2 (a) holds for A_{m-1}, $m > r$. By (5) the basis monomials of the induced linear space B_m is generated by $P_m = P_{m-1}^2 + x_m$. By inductive hypothesis P_{m-1} is spanned by all the monomials in the set

$$M_{m-1} = \{p_a = x_1^{a_1} x_2^{a_2} \ldots x_{m-1}^{a_{m-1}} : w(p_a) \leqslant 2^{m-r-1}\}.$$

$$[\text{since } w(p_a) = \sum_{k=1}^{r} a_k + 2a_{r+1} + \ldots + 2^{m-r-1} a_{m-1} \leqslant 2^{m-r-1}].$$

P_m is spanned by the products $p_a\, p_b$ ($p_a, p_b \in M_{m-1}$) and x_m.

Also

$$w(p_a p_b) = w(p_a) + w(p_b) \leqslant 2^{m-r-1} + 2^{m-r-1} = 2^{m-r}$$
$$w(x_m) = 2^{m-r}.$$

Hence P_m is spanned by all monomial p such that $w(p) \leqslant 2^{m-r}$. By induction Proposition 2 (a) holds for A_n. Hence the proposition.

PROOF OF PROPOSITION 2 (b) : Let $D(r, n, 2^{n-r})$ denote the number of solutions of the equation

$$a_1 + a_2 + \ldots + a_r + a_{r+1} + 2a_{r+2} + \ldots + 2^{n+r} a_{n+1} = 2^{n-r}.$$

It is known[3] that $D(r, n, 2^{n-r})$ is the coefficient of $t^{2^{n-r}}$ in the expansion of

$$\phi_{r,n}(t) = [(1-t)^r (1-t^2)(1-t^{2^2}) \ldots (1-t^{2^{n-r}})]^{-1}$$

in powers of t. We have

$$\phi_{(r+1),(n+1)}(t) = (1-t)^{-1} \phi_{r,n}(t)$$

$$= (1+t+t^2+\ldots)[D(r,n,0) + D(r,n,1)t + \ldots + D(r,n,2^{n-r})t^{2^{n-r}} + \ldots].$$

Equating the coefficients of $t^{2^{n-r}}$ from both sides, we get

$$D(r+1, n+1, 2^{n-r}) = \sum_{k=0}^{2^{n-r}} D(r, n, k).$$

The right-hand side is the number of all solutions of the inequality (6) whereas the left-hand side is the number of all solutions of the equation (7). Hence the result.

Remark : When $r = 1$, A_n is a genetic algebra with restriction $c_0^2 = c_0$. Hence Propositions 2 (a) and 2 (b) obtained here are extensions of the corresponding results of Abraham [vide Propositions 1 and 2 of Art. 3 in Abraham[1]].

4. Further Extensions

(i) There are certain genetic situations in which $(1_b')$ holds for $k = t_1, t_2, \ldots, t_r$. For instance, in Sec. 1, (iii) is such a situation. In such cases the method in section 2 can be used to show that maxium value of dim B for a genetic algebra of dim $(n+1)$ is the number of solutions of the diophantine inequality.

$$a_1 + 2a_2 + \ldots + 2^{t_1-1}(a_{t_1-1} + a_{t_1}) + \ldots + 2^{t_r-r}(a_{t_r-1} + a_{t_r})$$

$$+ \ldots + 2^{n-r} a_n \leqslant 2^{n-r}.$$

(ii) The method in section 2 can be used to obtain the induced linear space B_r with respect to a (minimal) linearization of a transformation ψ_r defined by $\psi_r(x) = a_0 + a_1 x + \ldots + a_r x^r$, where x^r, where x^r is an rth principal power in a genetic algebra. We have established in Singh and Singh[9] that B_r is independent of r and coincides with the induced linear space B with respect to a quadratic transformation in a genetic algebra with non-zero multiplication constants.

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References

1. V. M. Abraham, *Proc. Lond. Math. Soc.* (3) **40** (1980), 346–63.
2. V. M. Abraham, *Proc. Lond. Math. Soc.* (3) **40** (1980), 364–84.
3. L. E. Dickson, *History of the Theory of Numbers*, Vol. II. Chelsea, New York, 1952 p. 134.
4. I. M. H. Etherington, *Proc. R. Soc. Edinb.* Sec. A, **59A** (1939), 242–58.
5. H. Gonshor, *Proc. Edinb. Math. Soc.* (2) **17** (1971), 289–98.
6. P. Holgate, *J. Lond. Math. Soc.* **42** (1967), 489–96.
7. R. D. Schafer, *Amer. J. Math.* **71** (1949), 121–35.
8. A. Wörz-Busekros, *Algebras in Genetics* (Lecture Notes in Biomathematics No. 36), Springer Verlag, New York, 1980.
9. M. K. Singh, and D.K. Singh, On the linearization of principal power map in a genetic algebra. Submitted for publication to *Indian J. pure appl. Math.*

Indian J. pure appl. Math., **19** (6) : 539–548, June 1988

D'ALEMBERT'S FUNCTIONAL EQUATION ON PRODUCTS OF TOPOLOGICAL GROUPS

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Let G be a multiplicative group. Equation

$$f(xy) + f(xy^{-1}) = 2f(x)f(y) \text{ for } x, y \in G \qquad \text{...(A)}$$

is called D'Alembert's Funetional Equation on G. It was completely solved by Cauchy for a real-valued continuous function f on the additive group $\mathbb{R}$ of real numbers with the usual topology (Aczel, 1966, pp. 117-20). In this paper, we solve (A) when f is a continuous complex-valued function on the direct product of a family of subgroups of $\mathbb{R}^n$, with the product topology.

1. INTRODUCTION

Kannappan[4] considered complex-valued functions on arbitrary G satisfying

$$f(xy) + f(xy^{-1}) = 2f(x)f(y) \text{ for all } x, y \text{ in } G \qquad \text{...(A)}$$

and an additional condition

$$f(xyz) = f(xzy) \text{ for all } x, y, z \text{ in } G. \qquad \text{...(B)}$$

In section 3, we prove some properties of functions satisfying (A) and (B). In section 4, we consider functions on products of topological groups, satisfying (A) and (B).

2. PRELIMINARIES

In this paper, the following notations and conventions are followed unless otherwise stated.

G denotes a multiplicative group with identity e. A function on G means a complex-valued function. 'Additive function' on G is a group homomorphism of G to the additive group of complex numbers. If H is a subgroup of a topological group G then the topology on H is the subspace topology. $\mathbb{C}^n$, $\mathbb{R}^n$ have their usual inner product space structures. $\mathbb{C}^*$ denotes the multiplicative group of non-zero complex numbers. A non-zero function f on G means that f is not identically zero. We say that f on G is a D'Alembert's function if it satisfies (A). For definitions we refer to Hewitt and Ross[2].

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Remark 2.1 : A function f on G satifying (A) is non-zero if and only if $f(e) = 1$.

Remark 2.2 : The set $W = \{z \in \mathbb{C}^* : | z-1 | < \sqrt{3}\}$ contains only one subgroup of $\mathbb{C}^*$ viz $\{1\}$.

In Remarks 2.3, 2.4 and 2.5 below, let f on G be a non-zero function satisfying (A) and (B). In the proof of Theorem 2 in Kannappan[4], Remarks 2.3 and 2.4 below have been proved.

Remark 2.3 : Let $f(x) = \pm 1$ for all $x \in G$. Then f is a homomorphism of G to the multiplicative group $\{-1, 1\}$ and $f(x) = (f(x) + [f(x)]^{-1})/2$ for all x in G.

Remark 2.4 : Suppose there is x_0 in G such that $f(x_0) \neq \pm 1$. Let $\beta^2 = (f(x_0))^2 - 1$. Then g defined by $g(x) = f(x) + (f(xx_0) - f(x) f(x_0))/\beta$ is a homomorphism of G to $\mathbb{C}^*$ with $f(x) = (g(x) + [g(x)]^{-1})/2$ for all x in G.

Remark 2.5 : Let $g = f$ if $f(x) = \pm 1$ for all x in G and let g be as in Remark 2.4 otherwise. Then g is a homomorphism of G to $\mathbb{C}^*$ such that $f(x) = (g(x) + [g(x)]^{-1})/2$ for all x in G.

3. Some Properties of D'Alembert's Functions

Theorem 3.1—A function f on G is non-zero, bounded and satisfies (A) and (B) if and only if there is a character χ of G such that $f(x) = \operatorname{Re} \chi(x)$ for all x in G.

If G is a topological group, theorem holds when function, character are replaced by their respective continuous versions.

Proof : Suppose f satisfies the hypothesis, then g defined as in Remark 2.5 is a homomorphism of G to $\mathbb{C}^*$. Since f is bounded so is g. Hence, $| g(x) | = 1$ for all $x \in G$. Thus g is a character of G. Since

$$f(x) = (g(x) + [g(x)]^{-1})/2 = (g(x) + \overline{g(x)})/2,$$

we see that $f = \operatorname{Re} g$. Converse is obvious.

If G is a topological group, f and g can be taken to be continuous.

Remark 3.2 : Ljubenova[5] has proved that O'Connor's Theorem 5 holds for any abelian topological group. It follows from Remark 2.1 and above Theorem that O'Connor's theorem holds when G is an abelian topological group and for non-abelian group if f satifies (B). Above argument shows that O'Connor's theorem follows from Kannappan's theorem.

Proposition 3.3—Let H be a subgroup of a locally campact Hausdorff abelian group G. Let f on H be continuous and satisfy (A). Then f has a continuous extension on G also satisfying (A) on G.

PROOF : If $f \equiv 0$ on H, the zero map on G is the obvious extension. So let $f \neq 0$. Then by Corollary 1 of Theorem 2 of Kannappan[4], there is a continuous homomorphism g of H to $\mathbb{C}^*$ such that $f(x) = (g(x) + [g(x)]^{-1})/2$ for all x in H. Let $\chi(x) = g(x)/|g(x)|$ and $\xi(x) = \log|g(x)|$ for x in H. Then χ is a character and ξ is a real character of H and $g(x) = \exp(\xi(x))\,\chi(x)$ for all $x \in H$. Note that χ and ξ are both uniformly continuous. Since $T = \{z \in \mathbb{C} : |z| = 1\}$, $\mathbb{R}$ are complete Hausdorff uniform spaces and H is dense in cl$H$ (closure of H), χ, ξ can be uniquely extended to continuous functions χ', ξ' over cl$H$. It is easy to verify that χ' is a character and ξ' is a real character of cl$H$. Further χ' and ξ' admit continuous extensions χ_1 and ξ_1 over G, where χ_1 is a character and ξ_1 is a real character of G (see Hewitt and Ross[3], pp. 380 and 391). Define g_1 and f_1 on G by $g_1(x) = \exp(\xi_1(x))\chi_1(x)$ and $f_1(x) = (g_1(x) + [g_1(x)]^{-1})/2$. Clearly f_1 is the required extension of f.

Remark 3.4 : Let H and G be as in Proposition 3.3. Let L be a continuous additive function on H. Then L has a continuous extension to G, which is an additive function on G.

PROOF : Apply the same technique as above extending L first to cl$H$ and then to G.

Proposition 3.5—The following are equivalent on G :

(i) For a non-zero function f on G satisfying (A) and (B), there exists an additive function L on G such that $f(x) = \cos L(x)$ for all x in G.

(ii) For any homomorphism g of G to $\mathbb{C}^*$, there is an additive function L on G such that $g(x) = \exp(iL(x))$ for all $x \in G$.

(iii) For any character χ of G, there is a real character ξ of G such that $\chi(x) = \exp(i\,\xi(x))$ for all x in G.

If G is a topological group, Proposition holds when function, homomorphism, character, real character are replaced by their respective continuous versions.

PROOF : (i) $\Rightarrow$ (ii) : Let g be a homomorphism of G to $\mathbb{C}^*$. Then $f(x) = (g(x) + [g(x)]^{-1})/2$, $x \in G$ satisfies (A) and (B). So by (i), there is an additive function H on G such that $f(x) = \cos H(x) = (\exp(iH(x)) + \exp(-iH(x)))/2$ for $x \in G$. Hence, by Theorem 3 of Kannappan[4], $g(x) = \exp(iH(x))$ or $g(x) = \exp(-iH(x)$ for all x in G. So take $L = H$ or $-H$.

(ii) $\Rightarrow$ (iii) : Given a character χ of G, by (ii) there is an additive function L on G such that $\chi(x) = \exp(iL(x))$ for all x in G. Clearly L is a real character since $\mathrm{Im}\, L(x) = 0$, $|\chi(x)|$ being 1 for all x.

(iii) $\Rightarrow$ (i) : Suppose f satisfies (A) and (B) and is non-zero. By Theorem 2 of Kannappan[4], there is a a homomorphism g of G to $\mathbb{C}^*$ such that $f(x) = (g(x) + [g(x)]^{-1})/2$ for all x in G. Let $\chi(x) = g(x)/|g(x)|$ and $\phi(x) = \log|g(x)|$ for

$x \in G$. Then χ is a character, ϕ is a real character of G and $g(x) = \exp(\phi(x))\chi(x)$. By (iii), there is a real character ξ of G such that $\chi(x) = \exp(i\xi(x))$. So $g(x) = \exp(\phi(x) + i\xi(x))$. Let $L(x) = -i(\phi(x) + i\xi(x))$, $x \in G$. Clearly L is as stated in (i).

If G is a topological group, each of functions involved above can be taken to be continuous.

Theorem 3.6—Let f be a non-zero, real-valued function on G satisfying (A) and (B). Then :

(i) f is bounded if and only if $f(x) = \operatorname{Re}\chi(x)$ for some character χ of G.

(ii) f is unbounded if and only if $f(x) = \phi(x)\cosh\xi(x)$ where ϕ is a homomorphism of G to the multiplicative group $\{1, -1\}$ and ξ is a non-trivial real character of G.

If G is a topological group, theorem holds when function, homomorphism, character, real character are replaced by their respective continuous versions.

PROOF : (i) is Theorem 3.1.

(ii) $f(x) = (g(x) + [g(x)]^{-1})/2$, where is a homomorphism of G to $\mathbb{C}^*$. Since $f(x)$ is real, for every x, $|g(x)| = 1$ or $g(x)$ is real. So, range of g is contained in $X = \{z : |z| = 1\} \cup \{\text{non-zero reals}\}$. Since g is a homomorphism, range of g is a subgroup of $\mathbb{C}^*$ contained in X. But then it has to be either contained in $\{z : |z| = 1\}$ or $\{\text{non-zero reals}\}$.

If $g(G) \subset \{z : |z| = 1\}$ then g is character of G and hence for any $x \in G$, $f(x) = \operatorname{Re} g(x)$. So f is bounded, which is a contradiction to f is unbounded. Hence $g(G)$ must be contained in {non-zero reals} and $|g(x)| \neq 1$ for at least one $x \in G$. So g is a non-trivial homomorphism of G to the multiplicative group of non-zero reals. Let $\phi(x) = g(x)/|g(x)|$ and $\xi(x) = \log|g(x)|$ for $x \in G$. Then ϕ is a homomorphism of G to $\{-1, 1\}$, ξ is a non-trivial real character of G and $g(x) = \exp(\xi(x))\phi(x)$. Hence for any $x \in G$,

$$f(x) = (\exp(\xi(x))\phi(x) + [\exp(\xi(x))\phi(x)]^{-1})/2$$
$$= \phi(x)(\exp(\xi(x)) + [\exp(\xi(x))]^{-1})/2 = \phi(x)\cosh\xi(x).$$

The converse is obvious.

If G is a topological group and f is continuous, then by Corollary 1 of Theorem 2 of Kannappan[4] there is a continuous homomorphism g of G to $\mathbb{C}^*$ such that $f(x) = (g(x) + [g(x)]^{-1})/2$ for all $x \in G$. Hence, $\phi(x) = g(x)/|g(x)|$ and $\xi(x) = \log|g(x)|$ are continuous on G.

Remark 3.7 : If G is such that $G^2 = G$ and if ϕ is a homomorphism from G to the multiplicative group $\{1, -1\}$, then $\phi \equiv 1$.

D'ALEMBERT'S FUNCTION ON PRODUCTS OF TOPOLOGICAL GROUPS

In what follows, if $\{G_j : j \in J\}$ is a family of (topological) groups then $\Pi\, G_j$ denotes the direct product (with the product topology) and $\Pi^*\, G_j$ denotes the weak direct product (with the topology induced as a subspace of the direct product $\Pi\, G_j$ with the box topology) unless otherwise stated. For a summation or product that follows in this section index set is J unless otherwise stated.

We identify each G_j with the subgroup of $\Pi\, G_j$ formed by the x's all of whose coordinates except the jth are equal to the identity. Thus for each $j \in J$, $G_j \subset \Pi\, G_j$ and each x_j is an element of $\Pi\, G_j$. And similarly for $\Pi^*\, G_j$.

Theorem 4.1—Let $G = \Pi\, G_j$, where each $G_j, j \in J$ is a topological group. Let f on G be non-zero and satisfy (A) and (B). Then the following are equivalent :

(i) f is continuous.

(ii) $f(x) = (\Pi\, g_j(x_j) + [\Pi\, g_j(x_j)]^{-1})/2$ for all x $(= \langle x_j \rangle)$ in G, where $\{g_j : G_j \to \mathbb{C}^*, j \in J\}$ is a family of continuous homomorphisms in which only finitely many homomorphisms are non-trivial.

PROOF : (i) $\Rightarrow$ (ii) : By Corollary 1 of Theorem 2 of Kannappan[4], there is a continuous homomorphism g of G to $\mathbb{C}^*$ such that $f(x) = (g(x) + [g(x)]^{-1})/2$ for all x in G. Let g_j be the restriction of g to G_j. The $g_j, j \in J$ is a continuous homomorphism of G_j to $\mathbb{C}^*$. Hence, assertions (1) and (2) given below are sufficient to prove the result.

(1) g_j is identically equal to 1 for all but finite number of j's and

(2) $g(x) = \Pi\, g_j(x_j)$ for all x in G.

Let $W = \{z \in \mathbb{C}^* : |z - 1| < \sqrt{3}\}$ Since $g : G \to \mathbb{C}^*$ is continuous, $U = g^{-1}(W)$ is an open set in G. Hence by definition of product topology, all but finite number G_j's lie in U. Consequently, for all but finite number of j's, $g(G_j) \subset W$. Hence using Remark 2.2, we have, for all but finite number of j's, $g(G_j) = \{1\}$. Accordingly $g_j = g \mid G_j \equiv 1$ for all but finite number of j's. Assertion (2) can be easily checked.

(ii) $\Rightarrow$ (i) : Let $\{g_j : G_j \to \mathbb{C}^*, j \in J\}$ and f be as in (ii). Suppose g_j is identically equal to 1 for all j's except $j_1, j_2, \ldots, j_n$. Define $g : G \to \mathbb{C}^*$ by $g(x) = \Pi g_j(x_j)$, $x = \langle x_j \rangle \in G$. Since $\tilde{g}_k(x) = g_{j_k}(x_{j_k})$, $k = 1, \ldots, n$ are continuous and

$g(x) = \prod_{k=1}^{n} \tilde{g}_k(x)$, g is continuous. Hence f is continuous.

Definition 4.2—We say that a topological group G satisfies condition (C) if for every continuous character χ of G there is a continuous real character ξ of G such that $\chi(x) = \exp(i\, \xi(x))$ for all x in G.

Remark 4.3 : If H is a subgroup of $\mathbb{R}^n$, then H satisfies condition (C). If each G_j in Theorem 4.1 satisfies condition (C), then $g_j(x_j) = \exp(i L_j(x_j))$ where L_j is an additive homomorphism of G_j to $\mathbb{C}$. So we have Theorem 4.4.

Theorem 4.4— Let $G = \Pi G_j$, where each G_j, $j \in J$ is a topological group satisfying (C). Let f on G be non-zero and satisfy (A) and (B). Then the following are equivalent :

(i) f is continuous.

(ii) $f(x) = \cos [\Sigma L_j(x_j)]$ for all x in G, where $\{L_j : G_j \rightarrow \mathbb{C}, j \in J\}$ is a family of continuous additive functions in which only finitely many are non-zero.

(iii) $f(x) = \cos L(x)$, $x \in G$ for some continuous additive function L on G.

Lemma 4.5—Let H be a subgroup of the additive groug $\mathbb{R}^n$ (with the usual topology). An additive function L on H is continuous if and only if there is a vector a in $\mathbb{C}^n$ such that $L(x) = a.x$ for all x in H, where dot denotes the usual inner product in $\mathbb{C}^n$. If $L \equiv 0$ then we can take $a = 0$.

Proof : If L is continuous, by Remark 3.4, L has a continuous extension L' over $\mathbb{R}^n$ that is a continuous linear function of $\mathbb{R}^n$ to $\mathbb{C}$. Hence, there is a vector a in $\mathbb{C}^n$ such that $L'(x) = a.x$ for all x in $\mathbb{R}^n$. Consequently, $L(x) = a.x$ for all x in H. Converse is obvious.

Remark 4.6 : Let H be a subgroup of $\mathbb{R}^n$. Then using the technique of the above lemma it is easy to see that a real character ξ of H is continuous if and only if there is a vector a in $\mathbb{R}^n$ such that $\xi(x) = a.x$ for all x in H, where dot denotes the inner product in $\mathbb{R}^n$.

As an immediate consequence of Remark 4.3, Theorem 4.4 and Lemma 4.5 we have the main result of this paper.

Theorem 4.7— Let $G = \Pi G_j$, where each G_j, $j \in J$ is a subgroup of $\mathbb{R}^n$ (with the usual topology). Let f on G be non-zero and satisfy (A). Then f is continuous if and only if $f(x) = \cos(\Sigma a_j.x_j)$ for all x in G, where $\{a_j : j \in J\}$ is a set of vectors in $\mathbb{C}^n$, all but finitely many of which are zero and dot denotes the inner product in $\mathbb{C}^n$.

Remark 4.8 : Let $\{G_j : j \in J\}$ be a family of abelian topological groups. Let $\Pi^* G_j$ be the weak direct product with the subspace topology of ΠG_j with the product topology. Then $\Pi^* G_j$ is a dense subgroup of ΠG_j. Hence, using technique of Proposition 3.3 and Theorem 3 of Kannappan[4], it can be easily shown that a continuous function f on $\Pi^* G_j$ satisfying (A) has a unique continuous extension f^* to ΠG_j which satisfies (A). Using this it is easy to see that Theorem 4.7 is valid if we take G as the weak direct product $\Pi^* G_j$ of subgroups G_j of $\mathbb{R}^n$ with the topology induced as a subspace of ΠG_j with the product topology.

Now we consider functions satisfying (A) and (B) on the weak direct product of groups. We state the following result, which can be easily proved by using Theorem 2 of Kannappan[4].

Theorem 4.9—Let $G = \Pi^* G_j$, where each G_j, $j \in J$ is a group. Then f on G is non-zero and satisfies (A) and (B) if and only if there is a family $\{g_j : G \to \mathbb{C}^*, j \in J\}$ of homomorphisms such that $f(x) = (\Pi\, g_j(x_j) + [\Pi\, g_j(x_j)]^{-1}/2$ for all x in G.

Recall that, from now onwards, if $\{G_j : j \in J\}$ is a family of topological groups then $\Pi^* G_j$ denotes the weak direct product with the topology induced as a subspace of the direct product $\Pi\, G_j$ with the 'box-topology'.

Theorem 4.10—Let J be a countable index set and let $G = \Pi^* G_j$ where each G_j, $j \in J$ is a topological group. Let f on G be a non-zero function satisfying (A) and (B). Then the following are equivalent :

(i) f is continuous.

(ii) There is a family $\{g_j : G_j \to \mathbb{C}^*, j \in J\}$ of continuous homomorphisms such that $f(x) = (\Pi\, g_j(x_j) + [\Pi\, g_j(x_j)]^{-1})/2$ for all x in G.

(iii) There is a continuous homomorphism g of G to $\mathbb{C}^*$ such that $f(x) = (g(x) + [g(x)]^{-1})/2$ for all x in G.

Proof : (i) $\Rightarrow$ (ii) : By Corollary 1 of Theorem 2 of Kannappan[4], there is a continuous homomorphism g of G to $\mathbb{C}^*$ such that $f(x) = (g(x) + [g(x)]^{-1})/2$ for all $x \in G$. Since g is a continuous homomorphism of G to $\mathbb{C}^*$, the restriction map g_j of g to G_j, $j \in J$ is a continuous homomorphism of G_j to $\mathbb{C}^*$ and $g(x) = \Pi\, g_j(x_j)$ for all $x \in G$. Hence, $\{g_j = g \mid G_j : j \in J\}$ is the required family.

(ii) $\Rightarrow$ (iii) : Let $\{g_j : G_j \to \mathbb{C}^*, j \in J\}$ and f be as in (ii). Then $g(x) = \Pi\, g_j(x_j)$, $x \in G$ is a homomorphism of G to $\mathbb{C}^*$. Hence, it only remains to show that g is continuous.

If J is finite, then g is obviously continuous. So, let $J = \mathbb{N}$. Let $\epsilon > 0$ be given. Since g_j is continuous, $U_j = \{x_j : | g_j(x_j) - 1 | < \epsilon/2^j\}$ is a neighbourhood of e_j in G_j. Let $U = (\Pi\, U_j) \cap (\Pi^* G_j)$. Then U is a neighbourhood of the identity in the subspace topology of G induced as a subspace of $\Pi\, G_j$ with the box topology. If x is a point in U then for any $N \in \mathbb{N}$,

$$\begin{aligned} \left| \prod_1^N g_j(x_j) - 1 \right| &\leqslant \left| \prod_1^{N-1} g_j(x_j) \right| \left| g_N(x_N) - 1 \right| + \left| \prod_1^{N-2} g_j(x_j) \right| \\ &\quad \left\| g_{N-1}(x_{N-1}) \right| + \ldots + \left| g_1(x_1) - 1 \right| \\ &\leqslant \prod_1^{\infty} (1 + \epsilon/2^j) \left(\sum_1^{\infty} | g_j(x_j) - 1 | \right) \leqslant e^{\epsilon} \sum_1^{\infty} | g_j(x_j) - 1 | . \end{aligned}$$

Hence for any $x \in U$,

$$| g(x) - 1 | = | \Pi g_j(x_j) - 1 | \leqslant e^{\epsilon} \sum_1^{\infty} | g_j(x_j) - | < e^{\epsilon}.\epsilon$$

So, g is continuous at the identity and hence continuous on G.

(iii) $\Rightarrow$ (i) is obvious.

Proposition 4.11—Let J be a countable index set and let $G = \Pi^* G_j$, where each G_j, $j \in J$ is a topological group satisfying (C). Let f on G be a non-zero function satisfying (A) and (B). Then the following are equivalent :

(i) f is continuous.

(ii) There is a family $\{L_j : G_j \to \mathbb{C}, j \in J\}$ of continuous additive functions such that $f(x) = \cos [\Sigma L_j(x_j)]$ for all $x \in G$.

(iii) There is a continuous additive function L on G such that $f(x) = \cos L(x)$ for all x in G.

PROOF : Using Theorem 4.10 and Proposition 3.5 the result follows. Note that the sum $\Sigma L_j(x_j)$ has only finitely many non-zero terms as any element x in G has only finitely many coordinates different from identity.

Corollary 4.12—Let $\{G_j : j \in J\}$ be a countable family of subgroups of $\mathbb{R}^n$ (with the usual topology) and $G = \Pi^* G_j$. Let f on G be non-zero and satisfy (A). Then the following are equivalent :

(i) f is continuous.

(ii) $f(x) = \cos (\Sigma a_j.x_j)$ for all x in G, where $\{a_j : j \in J\}$ is a set of vectors in $\mathbb{C}^n$ and dot denotes the inner product in $\mathbb{C}^n$.

Remark 4.13 : In Theorems 4.1, 4.4, 4.7, 4.9, 4.10, Proposition 4.11 and Corollary 4.12, if f is also bounded then using Theorem 3.1 and methods used in respective results it is easy to show the following :

(i) In Theorems 4.1 and 4.10, g_j's are continuous characters and $f(x) = \text{Re}(\Pi g_j(x_j))$.

(ii) In Theorem 4.9, g_j's are characters and $f(x) = \text{Re}(\Pi g_j(x_j))$.

(iii) In Theorem 4.4 and Proposition 4.11, L_j, $j \in J$ and L are continuous real characters of G_j and G.

(iv) In Theorem 4.7 and Corollary 4 12, a_j's are vectors in $\mathbb{R}^n$ and dot denotes inner product in $\mathbb{R}^n$.

Remark 4.14 : In Theorem 4.10, Proposition 4.11 and Corollary 4.12 if the index set J is uncountable then (i) $\Rightarrow$ (ii) but (ii) $\not\Rightarrow$ (i). This is shown by the following :

Counter example : Let J be uncountable and let $G = \Pi^* G_j$, where eaeh $G_j = \mathbf{R}$ (with the usual topology). Let $\xi' : \mathbf{R} \to \mathbf{R}$ be defined by $\xi'(t) = \pi t$ for all $t \in \mathbf{R}$. For each $j \in J$, let $\xi_j = \xi'$ and $\chi_j = \exp(i \xi')$. Then $\{\xi_j : j \in J\}$ is a family of continuous real characters and $\{\chi_j : j \in J\}$ is a family of continuous characters. Define ξ, χ on G by $\xi(x) = \Sigma \xi_j(x_j)$ and $\chi(x) = \Pi \chi_j(x_j) = \exp(i \Sigma \xi_j(x_j))$.

Claim : ξ, χ are discontinuous.

Let $U = (\Pi U_j) \cap (\Pi^* G_j)$ be a basic neighbourhood for the identity in the subspace topology of G induced as a subspace of ΠG_j with the box topology. Then $\{U_j : j \in J\}$ is an uncountable family of neighbourhoods of zero. So there is a natural number n_0 such that $(-1/n_0, 1/n_0) \subset U_j$ for each j in some uncountable subset A of J. Let $a_1, a_2, \ldots, a_{n0+1}$ be any $n_0 + 1$ elements in A.

Let $x = \langle x_j \rangle$ in G be such that

$$x_j = 1/n_0 + 1 \text{ if } j = a_1, a_2, \ldots, a_{n0+1}$$

$$= 0 \text{ otherwise.}$$

Then $x \in U$ and $\xi(x) = \pi$ and $\chi(x) = -1$. Thus any basic neighbourhood of the identity in G contains x such that $\xi(x) = \pi$ and $\chi(x) = -1$. Hence ξ, χ are discontinuous. Since χ is discontinuous, $f = \operatorname{Re} \chi$ is also discontinuous. Thus (ii) $\not\Rightarrow$ (i).

Remark 4.15 : Kannappan[3] has shown that his condition

$$f(xyz) = f(xzy) \text{ for all } x, y, z \text{ in } G \tag{B}$$

is very restrictive and reduces the study of (A) on a non-abelian group G to the study of (A) over the abelian group G/G', where G' is the commutator subgroup of G. In view of this it is sufficient to study (A) on an abelian group. However, we study functions satisfying (A) and (B) on non-abelian groups because the proofs of the results proved in this paper are same in both the cases. Also we have advantage of studying (A) on the original group instead of studying it on the quotient group.

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References

1. J. Aczel, *Lectures on Functional Equations and Their Applications*. Academic Press, New York, 1966.

2. E. Hewitt, and K. Ross, *Abstract Harmonic Analysis*, Vol. I. Springer-Verlag, Berlin, 1979.
3. Pl. Kannappan, *Mat. Vesnik* **8** (23) (1971), 317–19.
4. Pl. Kannappan, *Proc. Am. Math. Soc.* **19** (1968), 69–74.
5. E. Ljubenova, *Aeq. Math.* 22 (1981), 54–55.
6. T. O'Connor, *Aeq. Math.* **15** (1977), 235–38.

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ON THE LIMIT $\Gamma(yi)$ AS y TENDS TO INFINITY

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The primary purpose of this paper is to show how techniques and topics in applicable analysis are put together to obtain the result.

There is a plethora of literature on the gamma function; hence, an explanation for this paper is in order. The result, lim $\Gamma(iy)$ as $y \to \infty = 0$ is not the motivating factor for writing this paper although the result has an application for the calculation of the absolute value of factorials of complex numbers. The main result was given by Stieltjes[1] in 1889 but the technique used is not accessible to the average reader. Thus, the primary purpose of this paper is a nice form of exposition to show how techniques and topics in applicable analysis are put together to obtain the result.

Techniques and topics used (quite different than in the work of Stieltjes) follow this sequence : (a) the extraction of a factor from an infinite product, (b) the representation of an infinite product by two infinite products, (c) the examination of convergence of an infinite product by determining an upper bound for a sequence of partial products, (d) the use of Euler's product fo $\pi z/\sin \pi z$, (e) the use of hyperbolic functions and (f) the computation of $|z!|$. These techniques and topics should be in the repertory of the applied analyst.

In some texts[2,3] the following problem is given as an exercise:

Prove that

$$|\Gamma(yi)|^2 = \pi/y \sinh \pi y.$$

The student is expected to use the reflection formula for the gamma function

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

However, the reflection formula. It is only fair to state, is not a simple matter to obtain. A detailed development of the above formula can be found in Ross[4]. Our starting point is Weierstrass's formula for the reciprocal of the gamma function

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n} \qquad \ldots(1)$$

where γ is Euler's constant. Let $z = x + yi$. We note that

$$\left| 1 + \frac{z}{n} \right| = \left| 1 + \frac{x}{n} + \frac{yi}{n} \right| = \left(\left(1 + \frac{x}{n} \right)^2 + \frac{y^2}{n^2} \right)^{1/2}$$

and that $| e^{\gamma(x+yi)} | = e^{\gamma x}$. Eqn. (1) can then be written as

$$\frac{1}{| \Gamma (x + yi) |} = (x^2 + y^2)^{1/2} e^{\gamma x} \prod_{n=1}^{\infty} \left(\left(1 + \frac{x}{n} \right)^2 + \frac{y^2}{n^2} \right)^{1/2} e^{-x/n}. \qquad \ldots (2)$$

The factor $\left(1 + \frac{x}{n} \right)$ is taken out of the infinite product above. Motivation for this step is explained later. The infinite product in eqn. (2) can be represented by the product of two infinite products :

$$\prod_{n=1}^{\infty} \left(1 + \frac{x}{n} \right) e^{-x/n} \prod_{n=1}^{\infty} \left(1 + \frac{y^2}{(x + n)^2} \right)^{1/2}. \qquad \ldots (3)$$

The first infinite product above is the infinite product in Weierstrass's definition eqn. (1) for $1/\Gamma(x)$ and its convergence is well-known. Of immediate concern is the convergence of the second infinite product. One can, of course, write out a few terms of the infinite product in eqn. (2) and a few terms of the products in eqn. (3).

From the identity that ensues, one can intuitively assume convergence of the second infinite product. However, we shall not rely on intuition; our recourse will be on the reliance on theorems of convergence of sequences. For convenience in writing, let $y^2/(x + n)^2$ be denoted by u_n.

We observe that for fixed x and y, the sequence of partial products $(1 + u_1)^{1/2}$, $(1 + u_1)^{1/2} (1 + u_2)^{1/2}$, $(1 + u_1)^{1/2} (1 + u_2)^{1/2} (1 + u_3)^{1/2}$, ... is a sequence of positive monotonically increasing functions of n and, accordingly approaches a limit if bounded. To determine if the sequence is bounded we write

$$1 + u_n \leq e^{u_n}$$

so that

$$(1 + u_n)^{1/2} \leq e^{u_n/2}.$$

Then

$$(1 + u_1)^{1/2} (1 + u_2)^{1/2} \ldots (1 + u_n)^{1/2} \leq e^{u_1/2} \cdot e^{u_2/2} \ldots e^{u_n/2}.$$

Now let n tend to infinity and we will have

$$\prod_{n=1}^{\infty} (1 + u_m)^{1/2} \leq \exp\left(\tfrac{1}{2} \sum_{1}^{\infty} u_n\right).$$

But $\sum_{1}^{\infty} u_n$ is convergent. Thus, the infinite product has an upper bound. Because the sequence of partial products is monotonic increasing and has an upper bound, the infinite product is convergent.

The next step is to put the infinite products in eqn. (3) in closed form. The first infinite product is the same product that appears in Weierstrass's definition in eqn. (1) so that

$$\prod_{n=1}^{\infty} \left(1 + \frac{x}{n} \right) e^{-x/n} = \frac{1}{x\, \Gamma(x)\, e^{\gamma x}} \cdot$$

When the expression on the right above is substituted into eqn. (2) we will have

$$\frac{1}{|\Gamma(x + yi)|} = \frac{(x^2 + y^2)^{1/2}}{\Gamma(x + 1)} \prod_{n=1}^{\infty} \left(1 + \frac{y^2}{(x + n)^2} \right)^{1/2} . \qquad \ldots(4)$$

The squaring of both sides and taking reciprocals gives

$$|\Gamma(x + yi)|^2 = \frac{(\Gamma(x + 1))^2}{x^2 + y^2} \prod_{n=1}^{\infty} \left(1 + \frac{y^2}{(x + n)^2} \right)^{-1} . \qquad \ldots(5)$$

An equivalent and simpler form of the above can be obtained if we start the product with $n = 0$:

$$|\Gamma(x + yi)|^2 = (\Gamma(x))^2 \prod_{n=0}^{\infty} \left(1 + \frac{y^2}{(x + n)^2} \right)^{-1} . \qquad \ldots(6)$$

Physicists use the above relation in calculations in the theory of beta ray decay. To evaluate $\Gamma(yi)$, let $x = 0$ in eqn. (5) and insert $i^2 = -1$ in the denominator of the infinite product getting

$$|\Gamma(yi)|^2 = \frac{1}{y^2} \prod_{n=1}^{\infty} \left(1 - \frac{(yi)^2}{n^2} \right)^{-1} \qquad \ldots(7)$$

The purpose of extracting the factor $\left(1 - \dfrac{x}{n} \right)$ in eqn. (2) is now clear. The

infinite product in eqn. (7) can be put into closed form. We recall Euler's infinite product for $\sin \pi z$ is

$$\frac{\pi z}{\sin \pi z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2} \right)^{-1}.$$

With $z = yi$, the product above is

$$\frac{\pi yi}{\sin \pi yi} = \frac{\pi yi}{i \sinh \pi y} = \frac{\pi y}{\sinh \pi y}. \qquad \ldots(8)$$

When the above is put into eqn. (7) we have the result

$$| \Gamma (yi) | = \sqrt{\frac{\pi}{y \sinh \pi y}}. \qquad \ldots(9)$$

If we treat ∞ as a number, we can write

$$\Gamma (\infty\, i) = 0.$$

The result (9) is the same for argument $-yi$, as can be seen in (8), because $\sinh (-\pi yi) = -\sinh \pi yi$.

One method to calculate $| z! |$ where $z = 2 + 4i$ is to use Legendre's duplication formula for $\Gamma (z)$ getting

$$\Gamma (3 + 4i) = 0.005225 \ldots - 0.17254 \ldots i.$$

Then

$$| (2 + 4i)! | = | \Gamma (3 + 4i) | = 0.1726 \ldots .$$

However, it is more expendient to use the result is eqn. (9). From elementary properties of the gamma function

$$| (2 + 4i)! | = | \Gamma (3 + 4i) | = | 2 + 4i \,||\, 1 + 4i \,||\, 4i \,||\, \Gamma (4i) | .$$

Equation (9) with $y = 4$ gives $| \Gamma (4i) = 0.00234049 \ldots$. We will have then

$$| (2 + 4i)! | = (\sqrt{20})\, (\sqrt{17})\, (4)\, (0.00234049) = 0.1726 \ldots .$$

A useful by-product of this analysis stems from eqn. (6) by first writing it as

$$\frac{| \Gamma (x + yi) |^2}{| \Gamma (x) |^2} = \prod_{n=1}^{\infty} \left(1 + \frac{y^2}{(x + n)^2} \right)^{-1}.$$

For fixed x and y the infinite product is less than unity. Thus, we can write

$$| \Gamma (x + yi) | \leqq | \Gamma (x) | .$$

REFERENCES

1. T. J. Stieltjes, *J. Math. Pure et Appliqués*, 4^e série, t. v. (1889), 426.
2. N. N. Lebedev, *Special Functions and Their Applications*. Dover Publications, (1972), p. 14.
3. Peter Henrici, *Applied and Camputational Complex Analysis*. Vol. 2, John Wiley and Sons, (1983), p. 35.
4. Bertram Ross, *Solved Problems in Analysis*, Dover Publications, (1971), p. 35.

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ON SYMMETRIZING A MATRIX

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A symmetrizer of an abritrary matrix A is a matrix X satisfying the matrix equations $XA = A^tX$ and $X = X^t$. A symmetrizer is useful in transforming an apparently nonsymmetric eigenvalue problem into a symmetric one. An algorithm to compute a matrix symmetrizer is presented and illustrated.

1. Introduction

A symmetrizer of an arbitrary square matrix A is a symmetric solution X that satisfies $XA = A^tX$, where t indicates the transpose. Symmetrizers reduce a nonsymmetric (real) eigenvalue problem (nep) into a symmetric eigenvalue problem (sep) which is relatively easy to solve and are useful in the stability problems of control theory[1]. These are also useful in the study of general matrices[9].

Transformation of nep to sep using a symmetrizer—Let A be a square nonsymmetric real matrix and X be a nonsingular symmetrizer of A. Then the nep $Ay = \lambda y$ can be reduced to an sep $Bx = \lambda x$ as follows. $Ay = \lambda y$ implies $XAy = \lambda Xy$. Since X is real symmetric, we have $X = P^t DP$, where P is orthogonal (i.e., $P^t = P^{-1}$) and $D = (d_{ii})$ is diagonal. Therefore, we have $PXAy = \lambda DPy$. Let $D_1 = (d^{\frac{1}{2}}_{ii})$ be a diagonal matrix such that $D_1^2 = D$. Then $PXAy = \lambda D_1^2\ Py$. $d_{ii} \neq 0$ for all i since X has nonzero eigenvalues (as X is nonsingular). Hence D_1^{-1} exists. Thus, $Bx = \lambda x$, where the symmetric matrix, real or complex,

$$B = D_1^{-1}\ PX\ AP^t\ D_1^{-1} \qquad \ldots(1)$$

and $x = D_1\ Py$. If X is positive definite then $d_{ii} > 0$ and consequently $d^{\frac{1}{2}}_{ii}$ is real. In this case B is real symmetric. If X has a negative eigenvalue then B is complex symmetric.

Symmetrizer to compute zeros of a real polynomial—A symmetrizer can be used to obtain the roots of a polynomial equation[5] The associated companion matrix[8] of the polynomial

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* supported by CSIR.

$$p_n(x) = \sum_{i=0}^{n} a_i x^i$$

where $a_n = 1$, and a_i $i = 0, 1, \ldots, n-1$ are real, is

$$A = \begin{bmatrix} 0 & 0 & \ldots & 0 & -a_0 \\ 1 & 0 & \ldots & 0 & -a_1 \\ \vdots & & & & \vdots \\ 0 & 0 & \ldots & 1 & -a_{n-1} \end{bmatrix}$$

whose eigenvalues are the zeros of $p_n(x)$. To obtain these eigenvalues, we compute an X such that $XA = A^t X$ and $X = X^t$ and then we compute the eigenvalues of the symmetric matrix $B = D_1^{-1} \;\; PXAP^t \;\; D_1^{-1}$, by a method, say, the Jacobi method[8,11] if the matrix X is positive definite. If the nonsingular X has one or more negative eigenvalues then a Cholesky-type decomposition[8,11] can be used to obtain the eigenvalues of B.

Existence of a symmetrizer—There exists a nonsingular symmetrizer for any square matrix[10]. Let A be a real nonsymmetric matrix and X be one of its symmetrizers. Then the following table depicts the scope of the symmetrizer X.

TABLE I

Scope of a symmetrizer

Eigenvalues of X	*Reduction of nep to sep*	*Utility*
All positive (X is positive definite)	Possible	Useful since sep is easier to solve (say, by Jacobi method)
One or more zero (X is singular)	Not possible	—
One or more negative and no zero	Possibe	Complex arithmetic needs to be used to solve sep (say, by Cholesky-type decomposition).

Since X is not unique, one can always compute another X by a different choice of some of the elements of X (see sec. 2) so that X is positive definite, a positive definite X does not, however, exist if the matrix A has complex eigenvalues.

2. Algorithm to Compute a Matrix Symmetrizer

Let $A = (a_{ij})$ be an $n \times n$ real nonsymmetric matrix. Also, let $X = (x_{ij})$ be a symmetrizer of A to be computed.

Step 1 : [Compute A_{ii}]

For $i := 1$ to $n - 1$ do

compute the $(n - i) \times (n - i + 1)$ matrix A_{ii},

where (k, t) th element of A_{ii}

$: = a_{i+l-1,k+l} - a_{ii}$ if $t + i - 1 = k + i$ else

$: = a_{i+l-1,k+l}$.

Step 2 : [Compute A_{ij} $j > i$]

For $i : = 1$ to $n - 1$ do

compute the $(n - i) \times (n - j + 1)$ matrix A_{ij}

where (k, t) th element of A_{ij}

$: = 0$ for $k : = 1$ to $j - i - 1$

$: = 0$ for $k : = j - i + 1$ to $n - i$ and $t \neq k - j + i + 1$

$: = - a_{j+t-1,l}$ for $k : = j - i$

$: = - a_{jl}$ for $k : = j - i + 1$ to $n - i$ and $t = k - j + i + 1$.

Step 3 : [Compute A_{ij} $j < i$]

For $i : = 2$ to $n - 1$ do

compute the $(n - i) \times (n - j + 1)$ matrix A_{ij},

where (k, t) th elememet of A_{ij}

$: = 0$ for $t : = 1$ to $i - j$

$: = 0$ for $t : = i - j + 2$ to $n - j + 1$ and $k \neq t + j - i - 1$

$: = a_{j,i+k}$ for $t : = i - j + 1$

$: = - a_{jl}$ for $t : = i - j + 2$ to $n - j + 1$ and $k = t + j - i - 1$.

Step 4 : [Obtain C]

Obtain $C = (A_{ij})$ $i = 1, 2, \ldots, n - 1, j = 1, 2, \ldots, n$.

Remark : C is an $(n^2 - n)/2 \times (n^2 + n)/2$ matrix.

Step 5 : [Solve $Cz = 0$]

Solve $Cz = 0$,

where

$$z = [x_{11}\ x_{12} \ldots x_{1n}\ x_{22}\ x_{23} \ldots x_{2n} \ldots x_{n-1,n-1}\ x_{n-1,n}\ x_{nn}]$$

as follows.

Step 5a : Choose, if none of the columns of C is zero, n elements $x_{11} = c \neq 0$, $x_{12} = x_{13} = \ldots = x_{1n} = 0$ or any n elements x_{ij} $(i \leq j)$ so that the resulting equations are consistent. If a column, which constitutes the coefficients of some x_{ij}, of C is zero then choose x_{ij} along with any other $n - 1$ elements of X so that the system is consistent and at least one of these $n - 1$ elements is nonzero to ensure nonsingularity of X. If k columns of C are zero then choose the corresponding x_{ij} along with any other $n - k$ elements of X (without disturbing consistency) so that at least one of these $n - k$ elements is nonzero to ensure nonsingularity of X. Let the resulting equations be $C' z' = b'$.

Step 5b : Use any method, say, the Gauss elimination method[8,11] to solve the equation — this gives X and terminate.

Proof of the algorithm—The proof steps 1 → 4 follows from the expansion and rearrangement of the equations $XA = A^t X$, $X^t = X$ and that of Step 5 (solving linear equations) is well-known.

2.1. *Stability of the algorithm*—The algorithm comprises (a) computing C, which is always stable, and (b) solving the linear equations $C' z' = b'$. Therefore, the numerical stability of the algorithm essentially depends on the degree of singularity of the matrix C' and on the stability of the method that is employed to solve $C' z' = b'$. The choice of the elements in Step 5 decides whether C' is near-singular, i.e., ill-conditioned with respect to the inverse, or not. There is, however, no easy way to know beforehand which will be a good choice. The selection of a stable method to solve linear equations usually poses no problem.

If the matrix whose symmetrizer $X = (x_{ij})$, where $x_1, x_2, \ldots, x_n$ are the rows of X, is required is lower Hessenberg and if we choose x_n of the form $[c\ 0\ 0\ \ldots\ 0]$, where $c \neq 0$ (see Sec. 4 also) then the algorithm is stable provided the Hessenberg matrix has the codiagonal nonzero. If any one element of the codiagonal is near-zero then the algorithm tends to become unstable unless some other appropriate choice is made.

3. Example

We compute a symmetrizer of the matrix $A = (a_{ij})$, where

$$A = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 3 & 0 \\ 1 & -1 & 2 \end{bmatrix}.$$

Here

$$C = \left[\begin{array}{ccc|cc|c} 1 & -2 & -1 & 0 & -1 & 0 \\ 0 & 0 & -3 & 0 & 0 & -1 \\ \hline 0 & 0 & -1 & 0 & -1 & 1 \end{array}\right]$$

and

$$z = [x_{11}\ x_{12}\ x_{13}\ x_{22}\ x_{23}\ x_{33}]'.$$

Choice 1—$x_{22} = 3$ (this choice is made since the corresponding column is null), $x_{11} = 2$, and $x_{12} = 1/2$. Any other choice for any three elements of X may be made preserving the consistency of the equations. Choice 1 gives the nonhomogeneous equations $C'z' = b'$, where

$$C' = \begin{bmatrix} -1 & -1 & 0 \\ -3 & 0 & -1 \\ -1 & -1 & 1 \end{bmatrix}, z' = \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix}, b' = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}.$$

Applying the Gauss reduction method with partial pivoting and then using the back substitution, we obtain $x_{33} = 1$, $x_{23} = 4/3$, $x_{13} = -1/3$. Hence the symmetrizer is

$$X = \begin{bmatrix} 2 & 1/2 & -1/3 \\ 1/2 & 3 & 4/3 \\ -1/3 & 4/3 & 1 \end{bmatrix}$$

which is positive definite. The symmetrizer transforms nep $Ay = \lambda y$ to sep $Bx = \lambda x$, where $B = D_1^{-1}\ PXAP^t\ D_1^{-1}$, $X = P^t DP$, and $D = D_1^2$ for some diagonal matrix D_1. From X, we compute using the Jacobi method

$$P = \begin{bmatrix} .1787430 & .8919905 & .4152155 \\ -.9434132 & .0355713 & .3297064 \\ .2793252 & -.4506525 & .8478737 \end{bmatrix},$$

$$D = \begin{bmatrix} 3.720851 & 0 & 0 \\ 0 & 2.097642 & 0 \\ 0 & 0 & 0.1815073 \end{bmatrix}.$$

Taking the positive square-root of the diagonal elements of D, we obtain D_1 (negative square-root can also be taken). Hence

$$B = \begin{bmatrix} 2.7547790 & -1.1969110 & -0.1342376 \\ -1.1969110 & 4.3387318 & -0.4728604 \\ -0.1342376 & -0.4728604 & 2.9302057 \end{bmatrix}.$$

It can be easily checked that the eigenvalues (computed by the Jacobi method) of B, viz., 5.042097, 2.999203, 1.982418 are approximately the same as those of the real nonsymmetric matrix A, which are 5, 3, 2.

Choice 2—$x_{22} = 1$, $x_{11} = 1$, $x_{12} = 0$. Solving the resulting nonhomogeneous equations, we obtain the nonsingular symmetrizer

$$X = \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 4/3 \\ -1/3 & 4/3 & 1 \end{bmatrix}.$$

having a negative eigenvalue, viz., -0.3743685. Hence $B = D_1^{-1} PXAP^t D_1^{-1}$

$$= \begin{bmatrix} 1.8348370 & -0.2159493 & i0.4159453 \\ -0.2159492 & 5.1176470 & i0.5438464 \\ i0.4159453 & i0.5438464 & 3.0475158 \end{bmatrix},$$

where

$$D = \begin{bmatrix} 2.3743690 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -0.3743685 \end{bmatrix}$$

and

$$P = \begin{bmatrix} 0.1714986 & -0.6859943 & -0.7071068 \\ 0.9701425 & 0.2425356 & -0.7304784E\text{-}20 \\ 0.1714986 & -0.6859943 & 0.7071068 \end{bmatrix}.$$

The eignvalues of the complex symmetric matrix B can be computed by a Cholesky-type decomposition; however, it can be easily checked that the eigenvalues of B are approximately the same as those of A, viz., 2, 3, 5.

All the foregoing computations have been carried out on DEC 1090-TOPS 10 computing system in single precision and 7 decimal digits are input.

4. Recursive Algorithm for a Lower Hessenberg Matrix

A simple and elegant recursive algorithm to compute a symmetrizer was suggested by Datta[2]. The algorithm is as follows. Let $A = (a_{ij})$ be an $n \times n$ lower Hessenberg matrix with nonzero codiagonal. Let $x_1, x_2, \ldots, x_n$ be the rows of a symmetrizer X to be computed.

Step 1 : Choose $x_n \neq 0$ arbitrarily. (To ensure nonsingularity of X, choose x_n of the form $[c\ 0\ 0 \ldots 0]$ where $c \neq 0$.)

Step 2 : Compute $x_{n-1}, x_{n-2}, \ldots x_1$ recursively from

$$x_i = (1/a_{i,i+1})\,(x_{i+1} A - a_{i+1,i+1}\, x_{i+1} - a_{i+2,i+1}\, x_{i+2} - \ldots - a_{n,i+1}\, x_n).$$

This algorithm is applicable only to lower Hessenberg matrices with nonzero codiagonal elements since each codiagonal element $a_{i,i+1}$ occurs in the denominator of Step 2 of the algorithm. However, any matrix can be reduced to a lower Hessenberg form by using similarity transformations involving Gauss-type elimination and this reduction is direct; thus, the eigenvalues remain invariant.

Derivation of the recursive algorithm—The foregoing recursive algorithm is a special case of the suggested general algorithm and is derived as follows. Consider the homogeneous equations $Cz = 0$ for the $n \times n$ nonsymetric real matrix $A = (a_{ij})$, where C and z are defined in Sec. 2. Substitute $a_{ij} = 0$ for $j = i + 2, i + 3, \ldots, n$ in C since A is assumed to be lower Hessenberg. Choose (as suggested by Datta) $x_{n1} = x_{1n} = c \neq 0$, $x_{n2} = x_{2n} = 0, \ldots, x_{nn} = 0$ in z and write the resulting linear equations $C' z' = b'$, where C' is a triangular matrix. Using back substitution, the recursive algorithm follows.

If A has any other special property then this can be incorporated in the general algorithm so that the general algorithm reduces to a simpler (special) one.

5. Conclusions

A general algorithm for computing a symmetrizer of an arbitray matrix A is suggested. A case where A is lower Hessenberg is shown to be a special case of the general algorithm. If the nonsymmetric matrix has any other special structure then this structure can be exploited to generate a special case of the algorithm. The advantage of obtaining a symmetrizer for computing the eigenvalues of a nonsymmetrix is yet to be fully investigated.

The computation of a symmetrizer involves a finite number of arithmetic operations and hence a symmetrizer can be computed exactly using finite-field transform techniques[3,4,6,7] superimposed on the (direct) method of solving linear equations. The need to have an exact symmetrizer arises when errors involved due to the use of real arithmetic is considerable.

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References

1. S. Barnett, Some topics in algebraic systems theory : a survey, *Recent Mathematical Developments in Control*, (ed. : D. J. Bell). Academic Press, New York, 1973, p. 323.
2. B. N. Datta, An algorithm for computing a symmetrizer of a Hessenberg matrix (unpublished).
3. R. T. Gregory, *Tech. Rep. CS-84-57, Computer Science Department, Univ. of Tennessee, Knoxville, TN 37916*, July 1984.
4. R. T. Gregory, and E. V. Krishnamurthy, *Methods and Applications of Error-free Computations*, Springer-Verlag; New York, 1984.
5. J. L. Howland, *Am. Math. Mon.* 68 (1961), 235.
6. E. V. Krishnamurthy, T. M. Rao, and K. Subramanian, *Proc. Indian Acad. Sci* **81 A** (1975), 58.
7. E. V. Krishnamurthy, T. M. Rao, and K. Subramanian, *Proc. Indian Acad. Sci.* **82 A** (1975), 165.

8. E. V. Krishnamurthy, and S. K. Sen, *Numerical Algorithms : Computations in Science and Engineering*, Affiliated East-West Press, New Delhi, 1986.
9. O. Taussky, *Linear Algebra and Appl.* 5 (1972), 147.
10. O. Tausaky, and H. Zassenhaus, *Pacific J. Math.* 9 (1959), 893.
11. J. H. Wilkinson, *Algebraic Eigenvalue Problem.* Oxford University Press, Oxford, 1965.

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CERTAIN EXPANSIONS ASSOCIATED WITH BASIC HYPERGEOMETRIC FUNCTIONS OF THREE VARIABLES

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In this paper, certain symbolic relations and expansions associated with basic hypergeometric functions of three variables have been established.

1. INTRODUCTION

Following Burchnall and Chaundy[1,2], Srivastava[4] has obtained single and double series expansions associated with hypergeometric functions of three variables by employing the inverse pair of operators :

$$\nabla_{x,y,z}(h) = \Gamma\begin{bmatrix} h, h+\delta_1+\delta_2+\delta_3 \\ h+\delta_1, h+\delta_2+\delta_3 \end{bmatrix} = \sum_{r,s\geqslant 0} \frac{(-\delta_1)_{r+s}(-\delta_2)_r(-\delta_3)_s}{(h)_{r+s}(1)_r(1)_s} \quad \ldots(1)$$

$$\Delta_{x,y,z}(h) = \Gamma\begin{bmatrix} h+\delta_1, h+\delta_2+\delta_3 \\ h, h+\delta_1+\delta_2+\delta_3 \end{bmatrix}$$

$$= \sum_{r,s\geqslant 0} \frac{(-\delta_1)_{r+s}(-\delta_2)_r(-\delta_3)_s}{(1-h-\delta_1-\delta_2-\delta_3)_{r+s}(1)_r(1)_s}, \quad \ldots(2)$$

where $\delta_1 = x\dfrac{\partial}{\partial x}$, $\delta_2 = y\dfrac{\partial}{\partial y}$, $\delta_3 = z\dfrac{\partial}{\partial z}$.

Here we shall establish certain symbolic relations and expansions of basic hypergeometric functions of three variables by making use of the following q-analogues of the inverse pair of symbolic operators (1) and (2) :

$$\nabla^{(q)}_{x,y,z}(h) = \pi\begin{bmatrix} h+\theta, h+\phi+\psi \\ h, h+\theta+\phi+\psi \end{bmatrix} \quad \ldots(3)$$

$$\Delta^{(q)}_{x,y,z}(h) = \pi\begin{bmatrix} h, h+\theta+\phi+\psi \\ h+\theta, h+\phi+\psi \end{bmatrix}. \quad \ldots(4)$$

The series equivalents of the operators in (3) and (4) are given by

$$\nabla^{(q)}_{x,y,z}(h) = \sum_{r,s\geqslant 0} \frac{(-\theta)_{r+s}\,(-\phi)_r\,(-\psi)_s\,h^{r+s}\,q^{r(\theta+\psi)+s(\theta+\psi)}}{(q)_r\,(q)_s\,(h)_{r+s}} \qquad \ldots(5)$$

and

$$\Delta^{(q)}_{x,y,z}(h) = \sum_{r,s\geqslant 0} \frac{(-\theta)_{r+s}\,(-\phi)_r\,(-\psi)_s\,q^{r+s-\psi r}}{(q)_r\,(q)_s\,(1-h-\theta-\phi-\psi)_{r+s}} \qquad \ldots(6)$$

where

$$\theta = x\frac{\partial}{\partial x},\ \phi = y\frac{\partial}{\partial y},\ \psi = z\frac{\partial}{\partial z},\ q^{\theta} = \exp(\theta \log q) \qquad \ldots(7)$$

and

$$(\theta)_n = (1-\theta)(1-\theta q)\ldots(1-\theta q^{n-1})\,;\ (\theta)_0 = 1.$$

The basic analogue of Srivastava's hypergeometric function of three variables[5] (p. 428) is defined as (see also Srivastava and Karlsson[6], p. 44) :

$$\Phi^{(3)}\left[\begin{matrix}(a)::(b);(c);(d):(e);(f);(g);\\ (a')::(b');(c');(d'):(e');(f');(g');\end{matrix}\ q;x,y,z\right]$$

$$= \sum_{m,n,p\geqslant 0} \frac{[(a)]_{m+n+p}\,[(b)]_{m+n}\,[(c)]_{n+p}\,[(d)]_{p+m}\,[(e)]_m\,[(f)]_n}{[(a')]_{m+n+p}\,[(b')]_{m+n}\,[(c')]_{n+p}\,[(d')]_{p+m}\,[(e')]_m\,[(f')]_n}$$

$$\times \frac{[(g)]_p\,x^m\,y^n\,z^p}{[(g')]_p\,(q)_m\,(q)_n\,(q)_p}. \qquad \ldots(8)$$

2. The Symbolic Relations

Employing the operators defined by (3) and (4), we have the following symbolic relations :

$$\Phi^{(3)}\left[\begin{matrix}a::-;-;-:b_1;b_2;b_3;\\ -::-;-;-;c_1;c_2;c_3;\end{matrix}\ q;x,y,z\right]$$

$$= \nabla^{(q)}_{x,y,z}(a).\,{}_2\Phi_1(a,b_1;c_1;x).\ \Phi^{(2)}(a,b_2,b_3;c_2,c_3;y,z). \qquad \ldots(8)$$

$$\Phi^{(3)}\left[\begin{matrix}a::-;-;-:b_1;b_2;b_3;\\ c::-;-;-;-;-;-;\end{matrix}\ q;x,y,z\right]$$

$$= \nabla^{(q)}_{x,y,z}(a).\,\Delta^{(q)}_{x,y,z}(c).\,{}_2\Phi_1(a,b_1;c;x).\ \Phi^{(1)}(a,b_2,b_3;c;y,z)$$

$$\ldots(9)$$

$$\Phi^{(3)}\left[\begin{matrix}-::-;-;-:a_1,b_1;a_2,b_2;a_3,b_3;\\ c::-;-;-:-;-;-;\end{matrix}\ q;x,y,z\right]$$

(*equation continued on p. 564*)

$$= \Delta^{(q)}_{x,y,z}\,(c)\cdot{}_2\Phi_1\,(a_1, b_1; c; x).\ \Phi^{(3)}\,(a_2; a_3; b_2, b_3\cdot c_3; y, z). \qquad \ldots(10)$$

3. Expansions

On making use of (5) in (8), we get

$$\Phi^{(3)}\left[\begin{matrix} a :: - ; - ; - : b_1 ; b_2 ; b_3 ; \\ - :: - ; - ; - : c_1 ; c_2 ; c_3 ; \end{matrix}\ q; x, y, z\right]$$

$$= \sum_{m,n,p=0}^{\infty} \sum_{r,s\geqslant 0} \frac{(q^{-m})_{r+s}\,(q^{-n})_r\,(q^{-p})_s\,a^{r+s}\,q^{r(m+n+p)+s(m+p)}}{(q)_r\,(q)_s\,(a)_{r+s}}$$

$$\times \frac{(a)_m\,(b_1)_m\,(a)_{n+p}\,(b_2)_n\,(b_3)_p\,x^m\,y^n\,z^p}{(c_1)_m\,(c_2)_n\,(c_3)_p\,(q)_m\,(q)_n\,(q)_p}$$

which (on replacing m by $m + r + s$, n by $n + r$ and p by $p + s$) gives the following series expansions, after some simplification :

$$\Phi^{(3)}\left[\begin{matrix} a :: - ; - ; - : b_1 ; b_2 ; b_3 ; \\ - :: - ; - ; - : c_1 ; c_2 ; c_3 ; \end{matrix}\ q; x, y, z\right]$$

$$= \sum_{r,s\geqslant 0} \frac{(a)_{r+s}\,(b_1)_{r+s}\,(b_2)_r\,(b_3)_s\,x^{r+s}\,y^r\,z^s\,a^{r+s}\,q^{2(r^2+s^2)+3sr}}{(q)_r\,(q)_s\,(c_1)_{r+s}\,(c_2)_r\,(c_3)_s}$$

$$\times\ \Phi^{(3)}\left[\begin{matrix} - :: - ; aq^{r+s} ; - : aq^{r+s}, b_1q^{r+s} : b_2q^r ; b_3q^s ; \\ - :: - ; - ; - : c_1q^{r+s}, c_2q^r ; c_3q^s ; \end{matrix}\ q ; xq^{r+s}, yq^r, zq^{r+s}\right]. \qquad \ldots(11)$$

Similarly, the symbolic relations (9) and (10) yield other series expansions for basic triple hypergeometric functions.

4. Special Expansions

Operating upon both sides of (7) by $\nabla^{(q)}_{x,y,z}\,(h)$ and then by $\Delta^{(q)}_{x,y,z}\,(k)$, we get,

$$\Phi^{(3)}\left[\begin{matrix} (a), k :: (b) ; c, h ; (d) : (e), h ; (f) ; (g) ; \\ (a'), h :: (b') : (c'), k ; (d') : (e'), k ; (f') ; (g') \end{matrix}\ q; x, y, z\right]$$

$$= \sum_{m,n,p=0}^{\infty} \sum_{r,s\geqslant 0} \frac{(q^{-m})_{r+s}\,(q^{-n})_r\,(q^{-p})_s\,k^{r+s}\,q^{r(m+n+p)+s(m+p)}}{(q)_r\,(q)_s\,(k)_{r+s}}$$

(equation continued on p. 565)

$$\times \frac{[(a)]_{m+n+p}\,[(b)]_{m+n}\,[(c)]_{n+p}\,[(d)]_{p+m}\,[(e)]_m\,[(f)]_n\,[(g)]_p\,(h)_m\,(h)_{n+p}\,x^m\,y^n\,z^p}{[(a')]_{m+n+p}\,[(b')]_{m+n}\,[(c')]_{n+p}\,[(d')]_{p+m}\,[(e')]_m\,[(f')]_n\,[(g')]_p\,(h)_{m+n+p}\,(q)_m\,(q)_n\,(q)_p}. \quad ...(12)$$

Now replacing m by $m+r+s$, n by $n+r$ and p by $p+s$ in the right hand-side of (12), we get, after some simplification, the following series expansion for basic hypergeometric funtions :

$$\Phi^{(3)}\left[\begin{matrix}(a), k :: (b) ; (c), h ; (d) : (e), h ; (f) ; (g) ; \\ (a'), h :: (b') ; (c'), k ; (d') : (e'), k ; (f') ; (g') ;\end{matrix}\; q; x, y, z\right]$$

$$= \sum_{r,s\geqslant 0} \frac{[(a)]_{2r+2s}\,[(b)]_{2r+s}\,[(c)]_{r+s}\,[(d)]_{r+2s}\,[(e)]_{s+r}}{[(a')]_{2r+2s}\,[(b')]_{2r+s})]\,[(c')]_{r+s}\,[(d,)]_{r+2s}\,[(e')]_{s+r}}$$

$$\times \frac{(h)_{r+s}\,(h)_{r+s}\,[(f)]_r\,[(g)]_s\,x^{r+s}\,y^r\,z^s\,k^{r+s}\,q^{2(r^2+s^2)+3sr}}{(h)_{2r+2s}\,(k)_{r+s}\,[(f)]_r\,[(g)]_s\,(q)_r\,(q)_s}$$

$$\times \Phi^{(3)}\left[\begin{matrix}(a)\,q^{2r+2s} :: (b)\,q^{2r+s} ; (c)\,q^{r+s}, hq^{r+s} ; (d)\,q^{r+2s} : \\ (a')\,q^{2r+2s}, hq^{2r+2s} :: (b')\,q^{2r+s} ; (c')\,q^{r+s} ; (d')\,q^{r+2s} :\end{matrix}\right.$$

$$\left.\begin{matrix}: (e)\,q^{r+s}, hq^{r+s} ; (f)\,q^r\,(g)\,q^s ; \\ : (e')\,q^{r+s} ; (f')\,q^r ; (g')\,q^s ;\end{matrix}\; q; xq^{r+s}, yq^r, zq^{r+s}\right]. \quad ...(13)$$

5. Special Cases

Taking $h = k$ in (13), we obtain the following peculiar type of expansion which has a free parameter k on the right only :

$$\Phi^{(3)}\left[\begin{matrix}(a) :: (b) ; (c) ; (d) : (e) ; (f) ; (g) ; \\ (a') :: (b') ; (c') ; (d') : (e') ; (f') ; (g') ;\end{matrix}\; q; x, y, z\right]$$

$$= \sum_{r,s\geqslant 0} \frac{(k)_{r+s}\,[(a)]_{2r+2s}\,[(c)]_{r+s}\,[(d)]_{r+2s}\,[(e)]_{r+s}}{(k)_{2r+2s}\,[(a')]_{2r+2s}\,[(c')]_{r+s}\,[(d')]_{r+2s}\,[(e')]_{r+s}}$$

$$\times \frac{[(f)]_r\,[(g)]_s\,[(b)]_{2r+s}\,x^{r+s}\,y^r\,z^s\,k^{r+s}\,q^{2(r^2+s^2)+3rs}}{[(f')]_r\,[(g')]_s\,[(b')]_{2r+s}\,(q)_r\,(q)_s}$$

$$\times \Phi^{(3)}\left[\begin{matrix}(a)\,q^{2r+2s} :: (b)\,q^{2r+s} ; (c)\,q^{r+s} ; (d)\,q^{r+2s}, kq^{r+s} : \\ (a')\,q^{2r+2s}, kq^{2r+2s} :: (b')\,q^{2r+s} ; (c')\,q^{r+s} ; (d')\,q^{r+2s} :\end{matrix}\right.$$

$$\left.\begin{matrix}: (e)\,q^{r+s}, kq^{r+s} : (f)\,q^r, (g)\,q^s ; \\ : (e')\,q^{r+s} ; (f')\,q^r ; (g')\,q^s ;\end{matrix}\; q ; xq^{r+s}, yq^r, zq^{r+s}\right]. \quad ...(14)$$

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REFERENCES

1. J. L. Burchnall, and T. W. Chaundy, *Quart. J. Math.* (*Oxford*) 11 (1940), 249-70.
2. J. L. Burchnall, and T. W. Chaundy, *Quart. J. Math.* (*Oxford*) 12 (1941), 112-28.
3. L. J. Slater. *Generalized Hypergeometric Functions*, Cambridge University Press, 1966.
4. H. M. Srivastava, *Rend. Circ. Mat. Palermo* (2) 16 (1967), 99-115.
5. H. M. Srivastava, *Proc. Camb phil. Soc.* 63 (1967), 425-29.
6. H. M. Srivastava, and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, John Wiley and Sons (Halsted Press), New York, 1985.

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SUBSTITUTION THEOREMS FOR INTEGRAL TRANSFORMS WITH SYMMETRIC KERNELS

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First a theorem revealing an interesting relationship between images and originals of a composite function of the form $k(x) F[g(x)]$ and the function $F(x)$ in two general integral transforms is established. Three other new theorems involving finite and infinite integral transforms are also obtained. By choosing the integral transforms T_1 and T_2 occurring in these theorems to be suitable integral transforms and on giving specific values to the functions $g(x)$, $k(x)$ therein, six specific theorems are obtained. Finally, an integral with the help of one of such theorems is evaluated.

1. INTRODUCTION

The present paper gives explicit expressions for the iteration of integral transforms with symmetric kernels.

Throughout this work we use the notation

$$h(p) = T\{f(x); p\} = \int_0^\infty k(p, x) f(x)\, dx \quad \ldots(1.1)$$

for the infinite integral transform, and the notation

$$h[p; (b, c)] = T\{f(x)[U(x - b) - U(x - c)]; p\}$$

$$= \int_b^c k(p, x) f(x)\, dx \quad \ldots(1.2)$$

for finite integral transform. In (1.1) and (1.2) the class of functions and the domain of p are so prescribed that these integrals always exist. Throughout this paper the kernel $k(p, x)$ is assumed to be symmetric i.e. $k(p, x) = k(x, p)$. $U(x - a)$ denotes the Heaviside (Unit Step) function.

2. Main Theorems

Theorem 1—If

$$h_1[p;(b,c)] = T_1\{k(x)\,h_2[g(x)]\,[U(x-b)-U(x-c)];p\} \qquad \ldots(2.1)$$

and

$$h_2(p^\sigma) = T_2\{f(x);p\} \qquad \ldots(2.2)$$

then

$$h_1[p;(b,c)] = \sigma \int_0^\infty \phi\{p,u;(\beta^{1/\sigma},\gamma^{1/\sigma})\}\,f(u)\,du \qquad \ldots(2.3)$$

where

$$\phi\{p,u;(\beta^{1/\sigma},\gamma^{1/\sigma})\} = T_2\{k_1[ph(y^\sigma)]\,k[h(y^\sigma)]\,y^{\sigma-1}$$
$$\times\, h'(y^\sigma)\,[U(y-\beta^{1/\sigma})-U(y-\gamma^{1/\sigma})];u\} \qquad \ldots(2.4)$$

provided that the integrals involved in eqns. (2.1)—(2.4) are absolutely convergent, p is independent of u, σ is a non-zero real number. $U(x-a)$ denotes the Heaviside (Unit Step) function, $k_1(p,x)$ and $k_2(p,x)$ are kernels of the transforms T_1 and T_2 respectively. The functions k, g, f and the inverse function $h = g^{-1}$ are single valued, analytic, real on $(0,\infty)$ and g is strictly monotonic on the subinterval (b,c) such that $g(b) = \beta$ and $g(c) = \gamma$.

Proof : If the substitution $x = h(y^\sigma)$ is made in (2.1), then

$$h_1[p;(b,c)] = \int_0^\infty k_1[ph(y^\sigma)]\,k[h(y^\sigma)]\,h_2[g\{h(y^\sigma)\}]$$
$$\times\,[U\{h(y^\sigma)-b\} - U\{h(y^\sigma)-c\}]\,\sigma\,y^{\sigma-1}\,h'(y^\sigma)\,dy$$

which easily yields after little simplification

$$h_1[p;(b,c)] = \sigma \int_{\beta^{1/\sigma}}^{\gamma^{1/\sigma}} k_1[ph(y^\sigma)]\,k[h(y^\sigma)]\,h_2(y^\sigma).\,y^{\sigma-1}\,h'(y^\sigma)\,dy). \qquad \ldots(2.5)$$

Now, noting the property that each kernel is symmetric, we have $k_2(y,u) = k_2(u,y)$ and so (2.2) yields the result

$$h_2(y^\sigma) = \int_0^\infty k_2(u,y)\,f(u)\,du. \qquad \ldots(2.6)$$

On substituting the value of $h_2(y^\sigma)$ from (2.6) in (2.5), we get

$$h_1[p;(b,c)] = \sigma \int_{\beta^{1/\sigma}}^{\gamma^{1/\sigma}} k_1[ph(y^\sigma)]\,k[h(y^\sigma)]\,\{\int_0^\infty k_2(u,y)\,f(u)\,du\}$$
$$\times\, y^{\sigma-1}\,h'(y^\sigma)\,dy.$$

Now, on interchanging the order of integrations in the above integral which is easily seen to be permissible under the conditions stated with the theorem, we get

$$h_1 [p; (b, c)] = \sigma \int_0^\infty \{ \int_{\beta^{1/\sigma}}^{\gamma^{1/\sigma}} k_2 (u, y) \, k_1 [ph (y^\sigma)] \, k [h (y^\sigma)] \, y^{\sigma-1} h' (y^\sigma) \, dy\} \times f(u) \, du \qquad ...(2.7)$$

which is (2.3).

If in Theorem 1, $b = 0$, $c \to \infty$ and the function g is so chosen that $g(0) = \beta$ and $g(\infty) = \gamma$, it reduces to the following interesting form :

Theorem 2—If

$$h_1 (p) = T_1 \{k (x) \, h_2 [g (x)]; p\} \qquad ...(2.8)$$

and

$$h_2 (p^\sigma) = T_2 \{ f(x); p\} \qquad ...(2.9)$$

then

$$h_1 (p) = \sigma \int_0^\infty \phi \{p, u; (\beta^{1/\sigma}, \gamma^{1/\sigma})\} f(u) \, du \qquad ...(2.10)$$

where

$$\phi \{p, u; (\beta^{1/\sigma}, \gamma^{1/\sigma})\} = T_2 \{k_1 [ph (y^\sigma)] \, k [h (y^\sigma)] \, y^{\sigma-1} h' (y^\sigma) \times [U (y - \beta^{1/\sigma}) - U (y - \gamma^{1/\sigma})]; u\} \qquad ...(2.11)$$

provided that conditions of validity directly obtainable from Theorem 1 are satisfied.

On the other hand, if in Theorem 1, $\beta = 0$, $\gamma \to \infty$ and the function g is so chosen that $g(b) = 0$ and $g(c) = \infty$, it takes the following form :

Theorem 3—If

$$h_1 [p; (b, c)] = T_1 \{k (x) \, h_2 [g (x)] \, [U (x - b) - U (x - c)]; p\} \qquad ...(2.12)$$

and

$$h_2 (p^\sigma) = T_2 \{ f(x); p\} \qquad ...(2.13)$$

then

$$h_1 (p; (b, c)] = \sigma \int_0^\infty \phi (p, u) \, f(u) \, du \qquad ...(2.14)$$

where

$$\phi (p, u) = T_2 \{k_1 [ph (y^\sigma)] \, k [h (y^\sigma)] \, y^{\sigma-1} h' (y^\sigma); u\} \qquad ...(2.15)$$

provided that conditions of validity directly obtainable from Theorem 1 are satisfied.

Also, if in Theorem 1, we take $b = \beta = 0$ and let $c \to \infty$, $\gamma \to \infty$ and the function g is so chosen that $g(0) = 0$ and $g(\infty) = \infty$ [or $g(0) = \infty$ and $g(\infty) = 0$], we arrive after a little simplification at the following interesting form of the theorem under appropriate conditions of validity easily obtainable from Theorem 1.

Theorem 4—If

$$h_1(p) = T_1\{k(x)\, h_2[g(x)];\, p\} \qquad \text{...(2.16)}$$

and

$$h_2(p^\sigma) = T_2\{f(x);\, p\} \qquad \text{...(2.17)}$$

then

$$h_1(p) = \sigma \int_0^\infty \phi(p, u) f(u)\, du \qquad \text{...(2.18)}$$

where

$$\phi(p, u) = T_2\{k_1[ph(y^\sigma)]\, k[h(y^\sigma)]\, y^{\sigma-1}\, h'(y^\sigma);\, u\}. \qquad \text{...(2.19)}$$

If in the above theorem we take $g(x) = x$, we at once arrive at a theorem given by Gupta[4]. Further if we take $\sigma = 1$ and $g(x) = x$ in the above theorem, we get a theorem obtained by Agrawal[1].

3. Specific Theorems

We next turn to find six specific theorems from the theorems obtained earlier. For doing so we give specific values to T_1, T_2, $g(x)$, $k(x)$ and σ occurring therein. Much of the computational work involved in obtaining these specific theorems is straightforward, but often lenghty, so we omit the same.

In Theorem 2, if we take $\sigma = 1$, $g(x) = e^{ax}$, $k(x) = e^{bx}$ and both transforms T_1 and T_2 to be the Laplace transforms, we easily get from (2.11) and the known result [Erdelyi[2], p. 137 (4)]

$$\phi\{p, u; (1, \infty)\} = \frac{1}{a}\, u^{p/a - b/a}\, \Gamma\left(\frac{b}{a} - \frac{p}{a}, u\right), \ \mathrm{Re}\, u > 0$$

resulting in the following specific theorem :

Theorem 2 (a)—If

$$h_1(p) = L\{e^{bx}\, h_2(e^{ax});\, p\} \qquad \text{...(3.1)}$$

and

$$h_2(p) = L\{f(x);\, p\} \qquad \text{...(3.2)}$$

then

$$h_1(p) = \frac{1}{a} \int_0^\infty u^{p/a - b/a}\, \Gamma\left(\frac{b}{a} - \frac{p}{a}, u\right) f(u)\, du \qquad \text{...(3.3)}$$

provided that the integrals involved in (3.1) to (3.3) are absolutely convergent and $\mathrm{Re}\, u > 0$.

In Theorem 3, if we take $\sigma = 1$, $g(x) = (x^2 - a^2)^{1/2}$, $k(x) = (x^2 - a^2)^{-1/4}$ and the transforms T_1 and T_2 to be the Laplace transform and the Hankel transform of order zero respectively, then from (2.15) and the known result [Erdelyi[3], p. 9 (24)], we easily get

$$\phi(p, u) = u^{1/2}(u^2 + p^2)^{-1/2} \exp[-a(u^2 + p^2)^{1/2}], \ \mathrm{Re}\, p > 0, \ \mathrm{Re}\, a > 0$$

resulting in the following specific theorem :

Theorem 3 (*a*)—If

$$h_1[p; (a, \infty)] = L\{(x^2 - a^2)^{-1/4} h_2[(x^2 - a^2)^{1/2}] U(x - a); p\} \quad \ldots(3.4)$$

and

$$h_2(p) = H_0\{f(x); p\} \quad \ldots(3.5)$$

then

$$h_1[p; (a, \infty)] = \int_0^\infty u^{1/2}(u^2 + p^2)^{-1/2} \exp[-a(u^2 + p^2)^{1/2}] f(u)\, du \quad \ldots(3.6)$$

provided that the integrals involved in (3.4) to (3.6) are absolutely convergent and $\mathrm{Re}\, p > 0$, $\mathrm{Re}\, a > 0$.

In Theorem 4, if we take $\sigma = 1$, $g(x) = x^2$, $k(x) = e^{-ax}$ and both the transforms T_1 and T_2 to be Laplace transforms, then from (2.19) and the known result [Erdelyi[2], p. 147 (33)], we easily get

$$\phi(p, u) = \tfrac{1}{2}\pi^{1/2} u^{-1/2} e^{(1/4u)(p+a)^2} \mathrm{Erfc}[\tfrac{1}{2}(p + a) u^{-1/2}],$$

$$\mathrm{Re}\, u > 0, \ |\arg(p + a)| < \pi/2$$

resulting in the following specific theorem :

Theorem 4 (*a*)—If

$$h_1(p) = L\{e^{-ax} h_2(x^2); p\} \quad \ldots(3.7)$$

and

$$h_2(p) = L\{f(x); p\} \quad \ldots(3.8)$$

then

$$h_1(p) = \tfrac{1}{2}\pi^{1/2} \int_0^\infty u^{-1/2} e^{\frac{1}{4u}(p+a)^2} \mathrm{Erfc}[\tfrac{1}{2}(p + a) u^{-1/2}] f(u)\, du \quad \ldots(3.9)$$

provided that the integrals involved in (3.7) to (3.9) are absolutely convergent and $\mathrm{Re}\, u > 0$, $|\arg(p + a)| < \pi/2$.

Again, in Theorem 4, if we take $\sigma = 1, g(x) = (x^2 + 2bx)^{1/2}, k(x) = x + b$ and the transforms T_1 and T_2 to be the Laplace Transform and Fourier sine transform respectively, then from (2.19) and the known result [Erdelyi[2], p. 75 (35)], we easily get

$$\phi(p, u) = e^{pb}\, pb^2\, u\, (u^2 + p^2)^{-1} K_2 [b\, (u^2 + p^2)^{1/2}], \ \mathrm{Re}\, p > 0, \ \mathrm{Re}\, b > 0$$

resulting in the following specific theorem :

Theorem 4 (*b*)—If

$$h_1(p) = L\, \{(x + b)\, h_2 [(x^2 + 2bx)^{1/2}];\, p\} \quad \ldots(3.10)$$

and

$$h_2(p) = F_s\, \{f(x);\, p\} \quad \ldots(3.11)$$

then

$$h_1(p) = pb^2\, e^{pb} \int_0^\infty u\, (u^2 + p^2)^{-1} K_2 [b\, (u^2 + p^2)^{1/2}]\, f(u)\, du \quad \ldots(3.12)$$

provided that the integrals involved in (3.10) to (3.12) are absolutely convergent and $\mathrm{Re}\, p > 0$, $\mathrm{Re}\, b > 0$.

Also, in Theorem 4, if we take $\sigma = 1, g(x) = a\,(e^{bx} - 1), k(x) = (e^{bx} - 1)^c$ and the transforms T_1 and T_2 to be the Laplace transform and the Stieltjes transform respectively, then from (2.19) and the known result [Erdelyi[3], p. 217 (9)], we easily get

$$\phi(p, u) = \frac{a^{\frac{p}{b} - c}}{b} \frac{\Gamma(c + 1)\, \Gamma\left(\frac{p}{b} - c + 1\right) u^{\frac{p}{b} - 1}}{\Gamma\left(\frac{p}{b} + 2\right) a^{\frac{p}{b} + 1}}$$

$$\times\, {}_2F_1\left(\frac{p}{b} + 1, c + 1, \frac{p}{b} + 2; 1 - \frac{u}{a}\right)$$

$$|\arg a| < \pi, \ 0 < \mathrm{Re}\,(c + 1) < \mathrm{Re}\left(\frac{p}{b} + 2\right)$$

resulting in the following specific theorem :

Theorem 4 (*c*)—If

$$h_1(p) = L\, \{(e^{bx} - 1)^c\, h_2 [a\, (e^{bx} - 1)];\, p\} \quad \ldots(3.13)$$

and

$$h_2(p) = S\, \{f(x);\, p\} \quad \ldots(3.14)$$

then

$$h_1(p) = \frac{1}{ba^{c+1}} \frac{\Gamma(c + 1)\, \Gamma\left(\frac{p}{b} - c + 1\right)}{\Gamma\left(\frac{p}{b} + 2\right)} \int_0^\infty u^{\frac{p}{b} - 1}$$

(*equation continued on p.* 573)

$$\times\ {}_2F_1\left(\frac{p}{b}+1, c+1, \frac{p}{b}+2; 1-\frac{u}{a}\right) f(u)\, du \quad \ldots(3.15)$$

provided that the integrals involved in (3.13) to (3.15) are absolutely convergent and $|\arg a| < \pi$, $0 < \mathrm{Re}\,(c+1) < \mathrm{Re}\,(p/b+2)$.

Also, in Theorem 4, if we take $\sigma = \frac{1}{2}$ $g(x) = (x^2 + 2bx)^{1/4}$, $k(x) = x^{-1/2}$ and the transforms T_1 and T_2 to be the Laplace transform and the Fourier cosine transform respectively, then from (2.19) and the known result [Erdelyi[2], p. 17 (29)], we easily get

$$\phi(p, u) = 2\, e^{pb} \left(\frac{\pi}{2}\right)^{1/2} [p + (p^2 + u^2)^{1/2}]^{1/2} (p^2 + u^2)^{-1/2}\, e^{-b(p^2+u^2)^{1/2}},$$

$$\mathrm{Re}\, p > 0,\ \mathrm{Re}\, b > 0,$$

resulting in the following specific theorem :

Theorem 4 (*d*)—If

$$h_1(p) = L\,\{x^{-1/2}\, h_2\,[(x^2 + 2bx)^{1/4}];\, p\} \quad \ldots(3.16)$$

and

$$h_2(p^{1/2}) = F_c\,\{f(x);\, p\} \quad \ldots(3.17)$$

then

$$h_1(p) = \left(\frac{\pi}{2}\right)^{1/2} e^{pb} \int_0^\infty [p + (p^2 + u^2)^{1/2}]^{1/2} (p^2 + u^2)^{-1/2}$$

$$\times\ \exp\{-\, b\,(p^2 + u^2)^{1/2}\}\, f(u)\, du \quad \ldots(3.18)$$

provided that the integrals involved in (3.16) to (3.18) are absolutely convergent and $\mathrm{Re}\, p > 0$, $\mathrm{Re}\, b > 0$.

A number of other specific theorems can also be obtained but we do not record them here for the lack of space.

4. Application

On making suitable choice of $f(x)$ in the specific theorems given above, we can evaluate a number of integrals. To illustrate, we evaluate an infinite integral by making use of the specific Theorem 2 (a). Thus, if we take $f(x) = x^{s-1}$ in Theorem 2 (a), then from (3.2) and the known result [Erdelyi[2], p. 133 (3)], we have

$$h_2(p) = \Gamma(s)\, p^{-s},\ \mathrm{Re}\, p > 0,\ \mathrm{Re}\, s > 0. \quad \ldots(4.1)$$

Further from (4.1), (3.1) and the known result [Erdelyi[2], p. 143 (1)], we easily get

$$h_1(p) = \Gamma(s)\,(p + as - b)^{-1},\ \mathrm{Re}\, p > -\,\mathrm{Re}\,(as - b). \quad \ldots(4.2)$$

Substituting the above value of $h_1(p)$ in (3.3), we get an interesting integral

$$\int_0^\infty u^{(p-b+sa-a)/a}\,\Gamma\left(\frac{b}{a}-\frac{p}{a},\,u\right)du = a\,\Gamma(s)\,(p+as-b)^{-1} \qquad \ldots(4.3)$$

where

$$\operatorname{Re}(p+as-b) > 0,\ \operatorname{Re} s > 0 \text{ and } \operatorname{Re} p > 0.$$

If we put $a = 1$, $b = 1$ and $1 - p = \alpha$ in the above integral, we get

$$\int_0^\infty u^{s-\alpha-1}\,\Gamma(\alpha, u)\,du = \Gamma(s)\,(s-\alpha)^{-1} \qquad \ldots(4.4)$$

where

$$\operatorname{Re}(s-\alpha) > 0,\ \operatorname{Re} s > 0.$$

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References

1. R. P. Agarwal, *Indian J. pure appl. Math.* **1** (1970), 537-51.
2. A. Erdelyi *et al.*, *'Tables of Integral Transforms' Vol. I. Mc-Graw Hill*, New York, 1954.
3. A. Erdelyi *et al.*, *'Tables of Integral Transforms' Vol. II. Mc-Graw Hill*, New York, 1954.
4. K. C. Gupta, *Int. J. Math. Math. Sci.* 5 (1982), 357–63.

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BRACHISTOCHRONE PROBLEM IN NONUNIFORM GRAVITY*

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The classical brachistochrone problem for which the solution turns out to be a cycloid, is based on the assumption that during the descent the gravity remains uniform. So the result will be valid only for a narrow region of space in which the variations in gravity can be neglected. The present paper gives a numerical study of brachistochrones taking into account the variation of gravity. The shapes of these curves have also been plotted for Earth's gravitational field and the times of arrivals at various points computed. The times of vertical descent and descent along straight lines have also been computed. Similar analysis will be useful in the field of an electric charge.

1. INTRODUCTION

The famous brachistochrone problem has a very interesting history.[1,2] This problem was raised by Galileo in 1638 and its 'solution' though not mathematically correct, was a source of inspiration for much work in the field. This work was initiated by John Bernoulli who challanged the mathematical world in June 1696 to solve the following problem :

'Given points A and B in a vertical plane to find a path down which a movable point, by virtue of its weight proceed from A to B in the shortest possible time'.

The above statement of the problem is followed by a paragraph in which he reassures his readers that the problem is very useful in mechanics and that it is not the straight line AB and that the solution curve is very well known to the geometers. He says he will show that this is so at the end of the year if no one else does. He extended this time limit to Easter of 1697. Finally, May 1697 issue of Acta Eruditorum appeared with John Bernoulli's solution. This issue also contained the solution of his elder brother James and a brief note by Leibniz saying that he had also solved the problem but since his solution was similar to that of Bernoulli brothers he would not reproduce it. Here it may be pointed out that some accounts are available which indicate that Newton had also given a solution of the problem.

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Leibniz had suggested that the problem be called 'Tachystoptotam' the problem of swiftest descent. Bernoulli, however, prefered to name name it the 'Brachistochrone' problem (from the Greek *brachistos* means shortest and *chronos* means time). Here it will not be possible to describe all the methods due to two Bernoulli brothers, Leibniz and Newton. One may see Goldstine[1] for details. The elegant method due to John Bernoulli was based on Fermat's principle of least time. Today, we have mathematical tools like calculus of variations available to us and the brachistochrone problem is merely a class work exercise, for which the solution is very well known to be a cycloid[2,3].

All the above discussion is valid only when the gravity is assumed to be uniform in the region in which the particle descends. The problem becomes much more difficult if the variation of gravity is taken into account. We have not come across many references dealing with this case. Smith[4] considers the problem through the Earth. He takes two points on the surface of Earth and assumes the movement of a particle through a tunnel dug between the two points through Earth's interior. It is well known that inside Earth the gravitational attraction is directly proportional to the distance from Earth's centre. Under these conditions, as shown by Smith, the complete integration of Euler-Lagrange equation is possible, though through a lengthy and tricky sequence of substitutions. The final result is remarkably interesting. The brachistochrone is found to be a *hypocycloid*. Smith also calculates the minimum transit time from San Diego to San Francisco. This is found to be about 12 minutes. Here it may be of interest to note that the length of the tunnel along brachistochrone would be about 685 miles as compared with 500 miles of the straight tunnel between them. The time of straight tunnel is found to be about 42 minutes.

As far as the brachistochrone problem outside Earth is concerned, we have not seen any papers. Here the gravity follows the inverse sqnare law and the exact integration of Euler-Lagrange equation is not possible. We have found a numerical solution to this problem in this paper. Plots of the brachistochrones have been obtained using CALCOMP plotter and the times of arrivals at various points computed. Comparision has been made with times taken along straight paths and times taken in a vertical fall.

2. Formulation of the Problem

Let a be the radius of Earth and let a particle move from a point P_s down a smooth curve to another point P_f as shown in Fig. 1. We have to find the brachistochrone joining P_s and P_f. Let O be the centre of Earth. Without any loss of generality, we can take the brachistochrone in the plane OP_sP_f. Taking $OP_s = r_s$ and $OP_f = r_f$, we should have $a < r_f \leqslant r_s$, otherwise, keeping in view the energy principle, the particle starting from rest at P_s will never reach P_f. Also let the angle P_sOP_f be α.

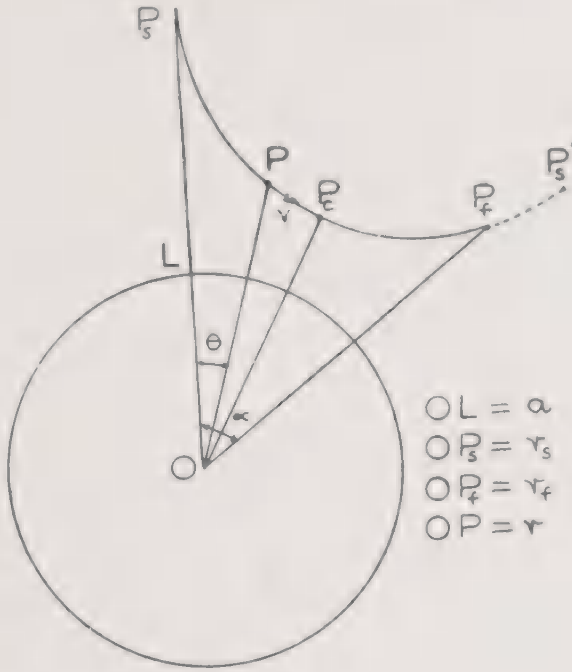

FIG. 1.

Using polar coordinates with initial line along OP_s, let the position of the moving point at time t be $P(r, \theta)$ and let its velocity there be v. If the particle starts from rest at P_s then the principle of conservation of energy gives

$$\frac{1}{2} mv^2 = GMm \left(\frac{1}{r} - \frac{1}{r_s} \right), \qquad \text{...(1)}$$

where m is the mass of the particle, M the mass of Earth and G the gravitational constant. Now if g be the acceleration due to gravity at the surface of Earth i.e. $g = MG/a^2$, we can write (1) as

$$v^2 = 2\, a^2 g \left(\frac{1}{r} - \frac{1}{r_s} \right). \qquad \text{...(2)}$$

Now using

$$v = ds/dt = [r^2 + (dr/d\theta)^2]^{1/2}\, d\theta/dt,$$

where t is time and s the arc length from P_s to P and integrating from P_s to P_f, the time taken is given by

$$t_0 = (2a^2 g)^{-1/2} \int_0^{\alpha} \left[\frac{r^2 + (dr/d\theta)^2}{r^{-1} - r_s^{-1}} \right]^{1/2} d\theta. \qquad \text{...(3)}$$

Let us introduce the following nondimensional quantities

$$R = r/a,\ R_s = r_s/a,\ R_f = r_f/a,\ T = t\,(2\,g/a)^{1/2}. \qquad \text{...(4)}$$

Then (3) gives

$$T_0 = \int_0^{\alpha} \left[\frac{R^2 + (dR/d\theta)^2}{R^{-1} - R_s^{-1}} \right]^{1/2} d\theta. \qquad \ldots(5)$$

Since the integrand does not contain θ explicitly one integration of the Euler-Lagrange equation associated with (5) leads to the following result after some simplification

$$\frac{dR}{d\theta} = \pm R \left(\frac{cR_s R^3 + R - R_s}{R_s - R} \right)^{1/2}. \qquad \ldots(6)$$

In the initial stages of motion, R will decrease with θ and so we take the negative sign. Then there comes a stage when $dR/d\theta = 0$ and we have the point of minimum R. After that R starts increasing symmetrically. The point where $dR/d\theta = 0$ will be called the *critical point* in the subsequent discussion. In Fig. 1, it has been denoted by P_c. The polar coordinates of P_c will be denoted by (R_c, θ_c). Note that if P_f lies on the part $P_s P_c$, the study of critical point will not be important for us. But if P_f is at P_c or beyond that its position will play an important role. The critical distance R_c can be found by solving the equation

$$f(R) = cR_s R^3 + R - R_s = 0. \qquad \ldots(7)$$

But this requires a prior knowledge of c, which cannot be determined unless (6) is integrated further and the two constants of integration found by satisfying the two boundary conditions. But some important information regarding the range of variation of c can be found even without doing so. From (6), it is easy to see that $c > 0$. Also $f(0) = -R_s$ and $f(R_s) = cR_s^4$. Further $df/dR = 3c R_s R^2 + 1 > 0$ in $(0, R_s)$. So there is precisely one real root of (7) in (O, R_s). It is again easy to see that it has no other real root. This real root, i.e. R_c should also satisfy the additional condition

$$R_c \geq 1, \qquad \ldots(8)$$

for those brachistochrones which are not obstructed by the Earth's surface. This demands

$$f(1) = cR_s + 1 - R_s \leqslant 0$$

or

$$0 < c \leqslant (R_s - 1) / R_s = c_0, \text{ say.} \qquad \ldots(9)$$

If the Earth is replaced by a point mass, it will not create any obstruction and c may lie anywhere between 0 and ∞.

For a given value of R_s, the critical value of c i.e. c_0 can be found from (9) and R_c can be computed from (7) for various values of c. For example, for $R_s = 2.0$, we

have $c_o = 0.5$ and the critical distances for some selected values of c are given in Table I.

TABLE I

Critical distances for $R_s = 2.0$

c	R_c	c	R_c	c	R_c	c	R_c
.005	1.928	.07	1.514	0.50	1.000	6.0	0.500
.01	1.869	.11	1.398	0.75	0.901	15.0	0.378
.02	1.776	.16	1.299	1.5	0.747	30.0	0.305
.03	1.703	.23	1.202	2.0	0.689	100.0	0.208
.05	1.595	.34	1.099	3.0	0.614	1000.0	0.098

Note that for $c > c_o = 0.5$, the full brachistochrones will be available only if the Earth is replaced by a point mass.

3. Computation of Brichistochrones

In this section we shall compute and plot the brachistochrones through P_s for several values of c. There will be a definite value of c for which it passes through the other point P_f. The determination of this value of c will be the subject matter of the next section.

Since the brachistochrones can be symmetrically extended beyond the crirical point, it is enough to discuss the paths from the starting point to the critical points for various values of c. Integrating (6) from P_s to P, we get

$$\theta = \int_R^{R_s} \frac{1}{R}\left(\frac{R_s - R}{cR_sR^3 + R - R_s}\right)^{1/2} dR. \qquad \ldots(10)$$

Since P can be any point on the arc P_sP_c, the lower limit can be varied from R_c to R_s and the corresponding value of θ can be found by evaluating the above integral numerically. Note that by virtue of (7), there is a singularity at the lower limit if it is R_c. So there will be difficulty in applying the usual quadrature methods. There are several special methods to handle such a situation[5]. We have found the critical angle θ_c by splitting the integral into two parts, i.e.

$$\theta_c = \int_{R_c}^{R_c+\epsilon} + \int_{R_c+\epsilon}^{R_s}$$

where ϵ is a suitable small positive number. The second integral is proper and can be evaluated by any numerical method. The first one can be approximated by

$$\frac{2}{R_c}\left[\frac{\epsilon\,(R_s - R_c)}{1 + 3\,cR_s\,R_c^4}\right]^{1/2}\left[1 + \frac{A}{3}\,\epsilon + \frac{B}{5}\,\epsilon^2 + O(\epsilon^3)\right] \qquad (11)$$

where

$$A = -\left[\frac{1}{R_c} + \frac{1}{2\,(R_s - R_c)} + \frac{3\lambda}{2}\;R_c\right] \qquad \ldots(12)$$

$$B = \frac{1}{R_c^2} - \frac{1}{8\,(R_s - R_c)^2} + \frac{27}{8}\;\lambda^2 R_c^2 + \lambda$$

$$+ \frac{1}{2\,R_c\,(R - R_c)} + \frac{3\,\lambda R_c}{4\,(R_s - R_c)}\,, \qquad \ldots(13)$$

$$\lambda = \frac{cR_s}{1 + 3\,cR_sR_c^2} = \frac{3\,R_c}{1 + 2\,cR_c^3}. \qquad \ldots(14)$$

These formulae were used to compute points on the brachistochrones for various values of c for a given R_s. The choice of ϵ and the strip size for evaluating the second integral depends on R_s. Table II gives the polar coordinates of the points on the brachistochrone for $R_s = 2.0$ and $c = 0.5$. The results are reported at equispaced radial distances upto the crirical point which in this case just happens to be situated on the Earth's surface. The various brachistochrones for this value of R_s and several values of c are plotted in Fig 2. Table II also gives the times of arrival at the various points. This will be discussed in section 5.

TABLE II

Coordinates and times of arrival

$(R_s = 2.0,\ c = 0.5)$

R	θ	*Time**	R	θ	*Time**
2.00	.0000	.0000	1.46	.0786	2.8582
1.94	.0018	.9763	1.40	.0993	3.0081
1.88	.0054	1.3756	1.34	.1243	3.1522
1.82	.0105	1.6788	1.28	.1549	3.2927
1.76	.0170	1.9319	1.22	.1929	3.4325
1.70	.0251	2.1528	1.16	.2417	3.5755
1.64	.0351	2.3511	1.10	.3077	3.7297
1.58	.0470	2.5323	1.04	.4102	3.9190
1.52	.0614	2.7004	1.00	.6060	4.2100

* 1 nondimensional unit of time (from (4))

$= (a/2\,g)^{1/2}$ secs, where $a = 6.37\times10^6\ m$, $g = 9.8$ m/sec^2.

$\doteq$ 570 secs or 9.5 minutes.

Note that in Fig. 2, the brachistochrones start intersecting Earth's surface for $> c_o = 0.5$. So they will exist only upto the point of intersection unless Earth is replaced by a point mass in which case, c may be given any value in $(0, \infty)$. All the curves will terminate at $R = 2$ where the velocity will become zero. Thus the particle

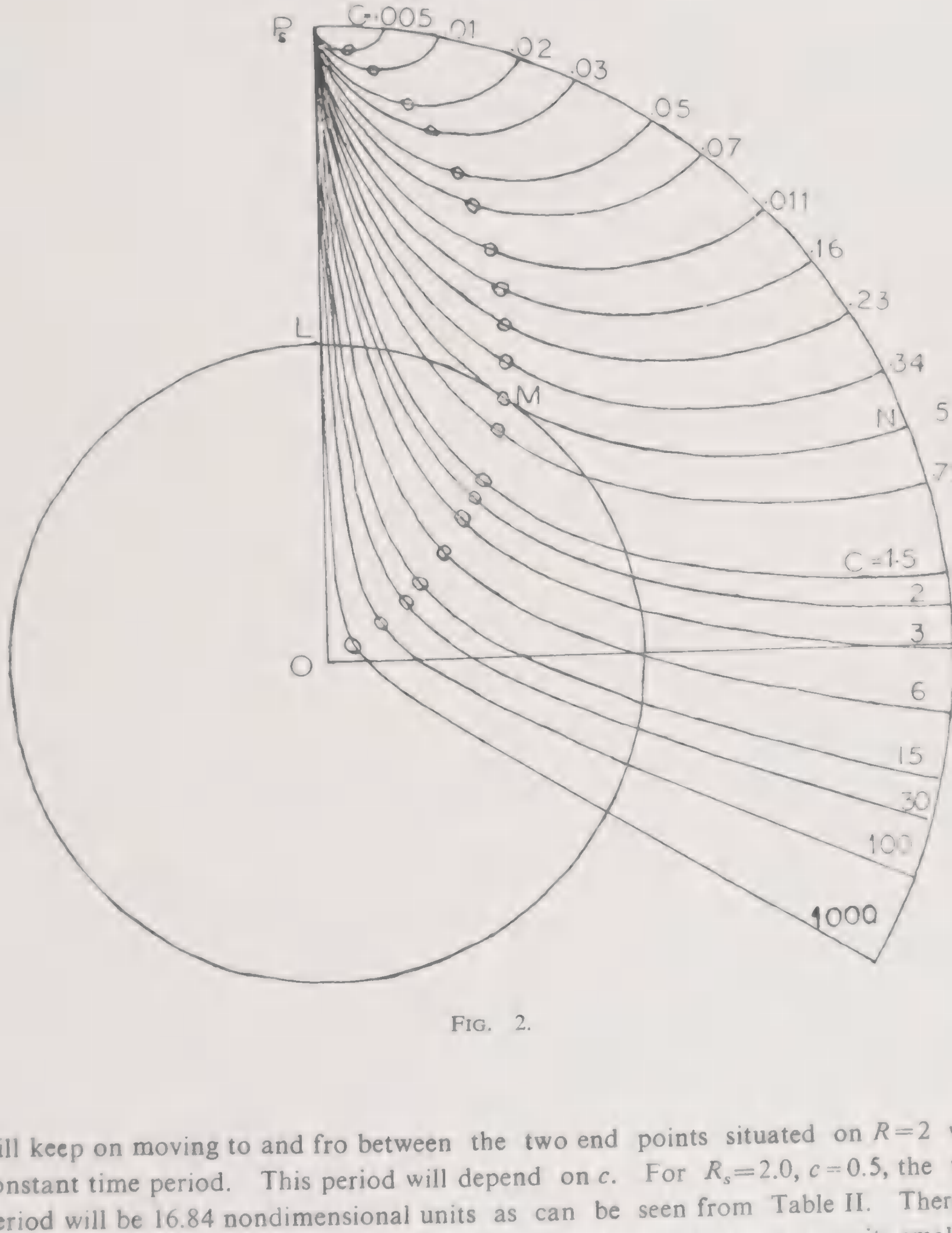

FIG. 2.

will keep on moving to and fro between the two end points situated on $R=2$ with constant time period. This period will depend on c. For $R_s=2.0$, $c=0.5$, the time period will be 16.84 nondimensional units as can be seen from Table II. There is one more observation to make. For small values of c, the curves are quite small in size and appear to be very close to a cyloid. It should indeed be so because in the space occupied by the curve the gravity will not vary appreciably and the problem will be reduced to the classical problem. However, as c is increased the curves go on departing more and more from the cycloidal shape as can be seen for $c=1000$. The critical points in Fig. 2 are shown circled. If necessary one can fit a suitable curve through these points from the numerical data.

4. Estimation of c When P_f is Given

In the previous section we have described the method of determining all the brachistochrones through a given starting point $P_s\,(R_s, 0)$. There will be one definite value of c for which one of these will pass through the other given point $P_f\,(R_f, \alpha)$. In this section, we discuss an iterative method to obtain c starting with a suitable initial approximation which may be obtained from Fig. 2.

From eqn. (10) we have

$$\alpha - \int_{R_f}^{R_s} \frac{1}{R}\left(\frac{R_s - R}{cR_sR^3 + R - R_s}\right)^{1/2} dR = \Phi\,(c),\ \text{say} = 0. \qquad \text{...(15)}$$

We have to solve this equation for c. We have applied Newton-Raphson method which gives

$$c_{k+1} = c_k - \frac{2\,\Phi\,(c_k)}{R_s\, I_k},\ k = 0, 1, 2, \ldots \qquad \text{...(16)}$$

where

$$I_k = \int_{R_f}^{R_s} \frac{R^2\,(R_s - R)^{1/2}\,dR}{(c_k\,R_sR^3 + R - R_s)^{3/2}}. \qquad \text{...(17)}$$

A computer program was prepared to carry out the above computations for given R_s, R_f and α. For $R_s = 2.$, $R_f = 1.$, $\alpha = 0.606$, it took 30 iterations with starting value $c_o = 0.3$ to converge to the value 0.5.

5. Computation of time upto a Point

The time taken by a particle from P_s to a point (R, θ) on the brachistochrone will be given by

$$T = \int_o^\theta \left(\frac{R^2 + R'^2}{R^{-1} - R_s^{-1}}\right)^{1/2} d\theta \qquad \text{...(18)}$$

where $R' = dR/d\theta$ is given by (6). Substituting this value and using the identity

$$cR_sR^3 + R - R_s = cR_s\,(R - R_c)\,(R^2 + R_c\,R + K),\ K = (cR_c)^{-1}, \qquad \text{...(19)}$$

we finally get

$$T = \sqrt{R_s} \int_R^{R_s} \frac{R^2\,dR}{[\,(R_s - R)\,(R - R_c)\,(R^2 + R_c\,R + K)\,]^{1/2}}. \qquad \text{...(20)}$$

The integrand has a singularity at the upper limit and also at lower limit when it is R_c. As already discussed there are various methods for handling such singularities

but computationally, we found the method of section 3 quite efficient numerically. We split the integral as follows

$$\sqrt{R_s}\left(\int_{R}^{R_s-\epsilon} + \int_{R_s-\epsilon}^{R_s}\right) = I_1 + I_2, \text{ say, if } R > R_c ,$$

and

$$\sqrt{R_s}\left(\int_{R_c}^{R_c+\epsilon} + \int_{R_s+\epsilon}^{R_s-\epsilon} + \int_{R_{-\epsilon}}^{R_s}\right) = I_3 + I_4 + I_2, \text{ if } R = R_c.$$

Proceeding as before, if can be shown that

$$I_2 = \frac{2R_s^{5/2}\ \epsilon^{1/2}}{[(R_s - R_c)(R_s^2 + R_s R_c + K)]^{1/2}}$$

$$\left[1 + \frac{A'}{3}\epsilon + \frac{B'}{5}\epsilon^2 + O(\epsilon^3)\right], \qquad \text{...(21)}$$

$$I_3 = \frac{2(R_s\,\epsilon)^{1/2}\ R_c^2}{[(R_s - R_c)(2\ R_c^2 + K)]^{1/2}}$$

$$\left[1 + \frac{A''}{3}\epsilon + \frac{B''}{5}\epsilon^2 + O(\epsilon^3)\right], \qquad \text{...(22)}$$

where

$$A' = -\frac{2}{R_s} + \frac{1}{2(R_s - R_c)} + \frac{p}{2}, \qquad \text{...(23)}$$

$$B' = R_s^{-2} + \frac{3}{8}[(R_s - R_c)^{-2} + p^2] - \frac{q}{2} + \frac{p}{R_s}$$

$$+ \left(\frac{p}{4} - \frac{1}{R_s}\right)(R_s - R_c)^{-1}, \qquad \text{...(24)}$$

$$p = (R_c + 2\ R_s)/(R_s^2 + R_s R_c + K), \qquad \text{...(25)}$$

$$q = 1/(R_s^2 + R_s R_c + K), \qquad \text{...(26)}$$

$$A'' = \frac{2}{R_c} + \frac{1}{2}(R_s - R_c)^{-1} - 3\lambda\ R_c/2, \qquad \text{...(27)}$$

$$B'' = R_c^{-2} + \frac{3}{8}[(R_s - R_c)^{-2} + 9\lambda^2 R_c^2] - \frac{7\lambda}{2}$$

$$+ \left(R_c^{-1} - \frac{\lambda R_c}{4}\right)(R_s - R_c)^{-1}. \qquad \text{...(28)}$$

The other integrals are proper and have been evaluated by using Simpson's rule. The numerical values of the times in arriving at various points for $R_8 = 2.0$ and $c = 0.5$ are reported in Table II in nondimensional units upto the critical point. Beyond that total times can be found by using symmetry of motion. It will be interesting to compare these times with those along straight paths and along vertical falls upto the same heights. This is done in the next section.

6. Time of Descent Along a Line

Let the points P_s and P_f be joined by a straight line. It is easy to see that for any point (R, θ) on this we have

$$R = \frac{R_s\, R_f \sin\alpha}{R_f \sin(\alpha-\theta) + R_s \sin\theta}\,. \qquad ...(29)$$

The time of descent from P_s to P_f along this line can be found by substituting R and $dR/d\theta$ obtained from this expression in (5). The resulting integral was not integrable in closed form. Since there is singularity at the starting point it is split into two parts as described above i.e. (i) from 0 to ϵ and then ϵ to α. The second is a proper integral and the Simpsons rule is applied to evaluate it. The first one is found to be

$$2\, R_s^{3/2}\ \epsilon^{1/2}(\mu+\mu^{-1})^{1/2}\left[1+\left(\frac{1}{4\mu}-2\,\mu\right)\frac{\epsilon}{3}+O(\epsilon^2)\right], \qquad ...(30)$$

where

$$\mu = \frac{R_s - R_f\cos\alpha}{R_f \sin\alpha}\,. \qquad ...(31)$$

A program was prepared to compute times for various values of R_s, R_f and α. Convergence was ensured by decreasing ϵ. With $R_s = 2.0$, $R_f = 1$, $\alpha = 0.606$ $\epsilon = 0.005$, the time was found to be 4.4162 units. The corresponding time along the brachistochrone as can be seen from Table II is 4.2100 units. As expected the time along the straight line is greater than along the brachistochrone.

It may be of interest to compute time taken by a particle in a free vertical fall from P_s to the surface of Earth. Integration of the equation of motion leads to the result

$$T = R_s^{3/2}\,[\cos^{-1}(R_s^{-1/2}) + R_s^{-1/2}(1 - R_s^{-2})^{-1/2}].$$

For $R_s = 2$, this gives

$$T = \frac{\sqrt{2}}{2}\,(\pi + \sqrt{6}) = 3.9535 \text{ units} \doteq 37.56 \text{ minutes}.$$

7. Conclusion

Although, the problem discussed here deals with brachistochrones in Earth's gravitational field, the method can be easily extended to other situations where the

field is variable. For example, one can take the gravitational field of two or more bodies and find the paths of shortest time. Since it will not be possible to construct brachistochrones of the dimension of Earth and verify the results of this paper experimentally, we suggest here an experiment which can be performed in the laboratory. Instead of a gravitating mass we can take an electrical charge and study the motion of another charged particle in its electric field. Since electric forces are much stronger and follow the same inverse square law the brachistochrones will be of similar shape. Similar experiments can be performed in the field of more than one charge.

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References

1. H.H. Goldstine, *A History of Calculus of Variations from the Seventeentn through the Nineteenth Century*. Springer-Verlag, 1980.
2. G. F. Simmons, *Differential Equations with Applications and Historical Notes*, Tata McGraw-Hill, 1972.
3. R. Kumar, M. Phil Dissertation submitted to Department of Mathematics, University of Roorkee, Sept., 1985.
4. D. R. Smith, *Variational Methods in Optimization*, Prentice Hall, 1974.
5. C. Froberg, *Introduction to Numerical Analysis*, Addison-Wesley Pub. Co., 1965.

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DIFFRACTION OF LOVE WAVES BY TWO PARALLEL PERFECTLY WEAK HALF PLANES

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We consider the diffraction of Love waves by two parallel perfectly weak half planes in a layer overlying a half space. The problem is formulated in terms of the Wiener-Hopf equations in the transformed plane. The transmitted waves are then calculated using the Wiener-Hopf procedure and inverse transforms.

1. INTRODUCTION

The diffraction of seismic waves by structural dicontinuities is of considerable importance in seismology because of the existence of such discontinuities in the Earth's crust. Exact analytical solutions of these problems are difficult to obtain even for simple geometries. de Hoop[2] presented a method based upon the Wiener-Hopf technique for the solution of body waves by a single perfectly rigid or perfectly weak half plane. Kazi[4] considered the diffraction of Love-waves by perfectly rigid and perfectly weak half planes lying in a surface layer overlying a half space. Recently, Asghar and Zaman[1] have considered the diffraction of Love waves by taking the rigid barrier to be of finite extension.

In this paper, we set up and solve the problem of diffraction of Love waves normally incident on two parallel perfectly weak (crack) half planes lying in a surface layer and parallel to the interface between the layer and the half space. The problem is formulated in terms of the two Wiener-Hopf equations and can be solved by the technique introduced by Jones[3]. The weak screens separate the layer into three loosely coupled layers. The transmitted waves in these three regions have been calculated analytically. As expected on physical grounds, it has been shown that the transmitted wave in each region satisfies the dispersion relation of the Love waves travelling in a layer of uniform thickness under similar boundary condition.

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2. Formulation of the Problem and the Wiener-Hopf Equations

We consider the diffraction of Love waves by two parallel weak half planes (cracks) lying in a layer of uniform thickness H over an elastic half space. The half space has a rigidity μ_1 and shear wave velocity β_1 and the layer has rigidity μ_2 and the shear wave velocity β_2. The coordinate system is chosen in such a way that the interface between the half space and the layered medium conicides with the xy plane, the z axis is directed into the half space and the two semi-infinite planes occupy $z = -h_1, x < 0$ and $z = -h_2, x < 0$. The free surface is $z = -H$. The geometry of the problem is shown in Fig. 1.

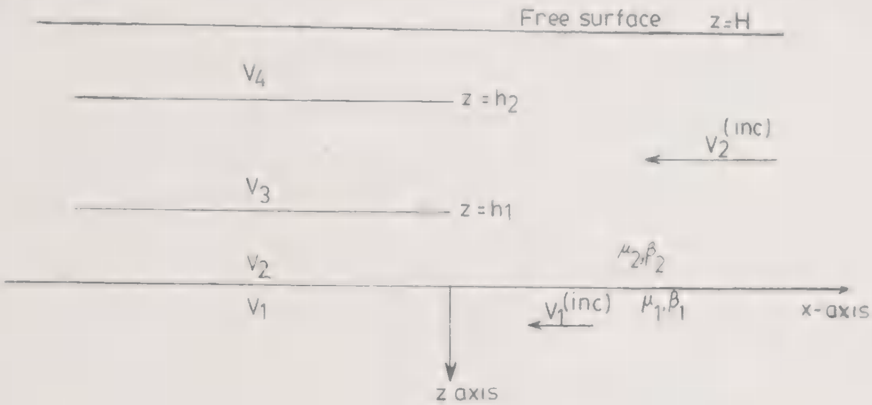

FIG. 1.

Suppressing the time dependence $e^{i\omega t}$, the incident Love waves of the Nth mode have the displacements :

$$V_1^{Inc} = A \cos(\sigma_{2N} H) \exp\{-\sigma_{1N} Z - iK_{1N}(x - x0)\}, Z > 0$$

$$V_2^{Inc} = A \cos\{(Z + H)\sigma_{2N}\} \exp\{-iK_{1N}(x - x0)\}, 0 > Z > -H \quad \ldots(1)$$

where

$$\sigma_{1N} = (K_{1N} - k_1^2)^{1/2}, \sigma_{2N} = (k_2^2 - K_{1N}^2)^{1/2}, \frac{\omega}{\beta_1}, \quad k_2 = \frac{\omega}{\beta_2} \quad \ldots(2)$$

and K_{1N} is the Nth root of the Love wave dispersion equation

$$\tan\{k_2^2 - K^2 H\} = \nu \frac{(K^2 - k_1^2)^{1/2}}{(k_2^2 - K^2)^{1/2}} \nu = \frac{\mu_1}{\mu_2} \quad \ldots(3)$$

corresponding to the layer thickness H. Moreover, $K_{1N} = \dfrac{\omega}{C_{1N}}$ $k = \omega/c$, where C_{1N} is the phase velocity of the Love waves of the Nth mode.

Let the total displacement field due to the presence of perfectly weak screens be written as

$$V_j^{total} = V^{(Inc)} + V_j, (j = 1, 2, 3, 4) \quad \ldots(4)$$

where V_j represents the diffracted displacements in the jth region as shown in Fig. 1. The geometry of the problem leads to the following boundary conditions :

(a) At $z = 0, \infty > x > -\infty$ ...(5a)

$$V_1 = V_2$$

$$\frac{\partial V_1}{\partial Z} = \frac{\partial V_2}{\partial Z}.$$

(b) For $x < 0$.

$$\left.\begin{array}{l} \text{At } Z = h_1 + 0, \dfrac{\partial V_2}{\partial Z} \\ \text{At } Z = -h_1 - 0, \dfrac{\partial V_3}{\partial Z} \\ \quad Z = -h_2 + 0, \\ \text{At } Z = -h_2 - 0, \dfrac{\partial V_4}{\partial Z} \end{array}\right\} = -\frac{\partial V_2^{inc}}{\partial Z} = A\sigma_{2N} \sin(\sigma_{2N}\delta_i) \exp - iK_{1N}(x - x0) \quad \ldots(5b)$$

$$\delta_i = H - h_i : i = 1, 2.$$

(c) At $z = -H, -\infty < x < \infty$

$$\frac{\partial V_4}{\partial Z} = 0 \quad \ldots(5c)$$

(d) At $z = -h_1, x > 0$

$$V_2 = V_3, \frac{\partial V_2}{\partial Z} = \frac{\partial V_3}{\partial Z} \quad \ldots(5d)$$

(e) At $z = -h_2, x > 0$

$$V_3 = V_4, \frac{\partial V_3}{\partial Z} = \frac{\partial V_4}{\partial Z}. \quad \ldots(5e)$$

The displacements V_j ($j = 1; 2, 3, 4$) satisfy the differential equations

$$\frac{\partial^2 V_j}{\partial x^2} + \frac{\partial^2 V_j}{\partial Z^2} + k_{1,2}^2 V_j = 0 \; (j = 1; j = 2, 3, 4) \quad \ldots(6,7)$$

where $k_i = \dfrac{\omega}{\beta i}$, $|k_1| < |k_2|$. The differential equations (6) and (7) together with the boundary conditions (5a) through (5e) constitute the boundary value problem.

With a little effort the above mixed boundary value problem can be written, in the Fourier transformed plane, in terms of the following two Wiener-Hopf equations:

$$\frac{1}{\sigma_2^2} \frac{\sigma_2 h}{\sinh \sigma_2 h} \frac{I(\alpha, h_2)}{I(\alpha, h_1)} \left\{ V'_{2+}(\alpha_1 - h_1) + V'_2 - (\alpha, h_1) \right\} - \frac{1}{\sigma_2^2} \frac{\sigma_2 h}{\sinh \sigma_2 h}$$

(equation continued on p 589)

$$\left\{V'_{4+}(\alpha, -h_2) + V'_{4-}(\alpha, -h_2)\right\} = h\,\{V_{3-}(\alpha, -h_1) - V_{2-}(\alpha, -h_1)\} \qquad \ldots(8)$$

$$\frac{1}{\sigma_2^2}\,\frac{\sigma_2 h}{\sinh \sigma_2 h}\left\{V'_{2+}(\alpha, -h_1) + V'_{2-} + (\alpha, h_1)\right\}$$

$$- \frac{1}{\sigma_2^2}\,\frac{\sigma_2 h}{\sinh \sigma_2 h}\,\frac{\sinh \sigma_2 \delta_1}{\sigma_2 \delta_1}\,\frac{\delta_1}{\delta_2}$$

$$\times \left\{V'_{4+}(\alpha, -h_2) + V'_{4-}(\alpha, -h_2)\right\} = \{V_{3-}(\alpha, -h_2) - V_{4-}(\alpha, h_2)\} \qquad \ldots(9)$$

where

$$\sigma_i^2 = (\alpha^2 - k_i^2)^{1/2};\; I(\alpha, h_i) = \sigma_2 \sin \sigma_2 h_i + \nu\sigma_1 \cosh \sigma_2 h_i$$

$$h = h_2 - h_1,\; \delta_l\; H - h_l.$$

In equations (8) and (9) the functions with subscript '+' are analytic in the domain $\text{Im}(\alpha + k_1) > 0$ and those with subscript '−' are analytic in the domain $\text{Im}(\alpha - k_1) < 0$.

3. Determination of the Wiener-Hopf Solution

The solution of the Wiener-Hopf equations (8) and (9) can be obtained by the usual procedure outlined by Noble[6]. Omitting the details of calculation, we can write the diffracted field in various regions as :

$$V_2(\alpha, Z) = -\frac{\sigma_2 \cosh \sigma_2 Z - \nu\sigma_1 \sinh \sigma_2 Z}{\sigma_2 \sinh \sigma_2 h_1 + \nu\sigma_1 \cosh \sigma_2 h_1}\,\frac{H_+(\alpha)}{T_+(\alpha)}\left(\frac{\alpha + k}{\alpha - k}\right)^{1/2}$$

$$\left[N_+(\alpha) - \frac{iA}{\sqrt{2\pi}}\,\frac{\sigma_2 N}{(\alpha - K_{1N})}\exp(iK_{1N}\, x0)\left\{\frac{T_+(K_{1N}) \sin \sigma_{2N} \delta_1}{(K_{1N} + k_2)\, H_+(K_{1N})}\right.\right.$$

$$\left.\left.- \sin \sigma_{2N} \delta_2 \left[P_+(\alpha) - P_+(K_{1N})\right]\right\}\right]. \qquad \ldots(10)$$

$$V_3(\alpha, Z) = \frac{\cosh \sigma_2 (Z + H)}{\sinh \sigma_2 H}\,\frac{H_+(\alpha)}{T_+(\alpha)}\left(\frac{\alpha + k_2}{\alpha - k_2}\right)^{1/2}$$

$$\left[N_+(\alpha) - \frac{iA}{2\pi}\,\frac{\sigma_{2N} \exp(ik_{1N}\, x0)}{(\alpha - K_{1N})} \times \frac{T_+(K_{1N}) \sin(\sigma_{2N})\,\delta_1}{(K_{1N} + k_2)\, H_+(K_{1N})}\right.$$

$$\left.- \sin(\sigma_{2N}\,\delta_2)\left[P_+(\alpha) - P_+(K_{1N})\right]\Big\}\right]$$

$$\frac{\cosh \sigma_2 (Z + h_1)\, H_+(\alpha)}{\sinh \sigma_2 h\; Y_+(\alpha)}\left(\frac{\alpha + k_2}{\alpha - k_2}\right)^{1/2}$$

(equation continued on p. 590)

$$\left[O_+(\alpha) - \frac{iA}{2\pi} \cdot \frac{\sigma_{2N} \exp(iK_{1N}\, x0)}{(\alpha - K_{1N})} \times \frac{Y_+(K_{1N}) \sin \sigma_{2N}\, \delta_2}{(K_{1N} + k_2)\, H_+(K_{1N})}\right.$$

$$\left.- \sin(\sigma_{2N}\, \delta_1)\, [W_+(\alpha) - W_+(K_{1N})] \Big\}\right]. \quad \ldots(11)$$

$$V_4(\alpha, Z) = \frac{\cosh \sigma_2 (Z + H)}{\sinh \sigma_2\, \delta_2} \frac{H_+(\alpha)}{Y_+(\alpha)} \left(\frac{\alpha + k}{\alpha - k}\right)^{1/2}$$

$$\left[O_+(\alpha) - \frac{iA}{2\pi} \frac{\sigma_{2N} \exp(iK_{1N}\, x0)}{(\alpha - K_{1N})} \times \left\{\frac{Y_+(K_{1N}) \sin(\sigma_{2N}\, \delta_1)}{(K_{1N} + k_2)\, H_+(K_{1N})}\right.\right.$$

$$\left.\left.- \sin(\sigma_{2N}\, \delta_1)\, [W_+(\alpha) - W_+(K_{1N})]\right\}\right]. \quad \ldots(12)$$

In obtaining these results, we have used the factorization

$$\frac{\sinh \sigma_2 h}{\sigma_2 h} = H_+(\alpha).\, H_-(\alpha)$$

$$\frac{I(\alpha, h_2)}{I(\alpha, h_1)} = T_+(\alpha).\, T_-(\alpha) \quad \ldots(13)$$

$$\frac{\sinh(\sigma_2\, \delta_1)}{\sigma_2\, \delta_1} \frac{\sigma_2\, \delta_2}{\sinh \sigma_2\, \delta_2} = Y_+(\alpha).\, Y_-(\alpha)$$

which have been explicitly obtained in the Appendix (A). Also, the splitting technique of Noble[6] is used to write the following additive decompositions (the explicity forms are given in Appendix B).

$$\frac{V'_{4+}(\alpha, -h_2)}{(\alpha + k_2)\, H_+(\alpha)\, T_-(\alpha)} = N_+(\alpha) + N_-(\alpha),$$

$$\frac{1}{(\alpha + k_2)\, H_+(\alpha)\, T_-(\alpha)} = P_+(\alpha) + P_-(\alpha) \quad \ldots(14)$$

$$\frac{V'_{2+}(\alpha, h_1)}{(\alpha + k_2)\, H_+(\alpha)\, Y_-(\alpha)} = O_+(\alpha) + O_-(\alpha)$$

$$\frac{1}{(\alpha + k_2)\, H_+(\alpha)\, Y_-(\alpha)} = W_+(\alpha) + W_-(\alpha).$$

4. The Transmitted Waves

We determine the transmitted waves in the three regions that are formed by the half planes in the layer. This can be done by taking inverse transforms. We do this for each region separately.

(*a*) *The region* $-h_1 < z < 0;\ x < 0$

The Fourier inversion formula gives

$$v_2(x, z) = \frac{1}{2\pi} \int_{c-\infty}^{c+\infty} \exp(-i\alpha x)\, V_2(\alpha, z)\, d\alpha \qquad \ldots(15)$$

where $V_2(\alpha, z)$ is given (10) and $\mathrm{Im}(k_1) > c > -\mathrm{Im}(k_1)$. For $x < 0$; $-h_1 < z < 0$, we can close the contour in the upper half plane. Lapwood[5] showed that the contributions to the surface waves come through the poles. The branch point contributions give rise to body waves which are of no interest to us for the present study. The integrand in eqn. (15) has simple poles at $\alpha = K_{1N}$ and at the zeros of $I(\alpha, h_1)$ located in the upper half plane. Using the relations (2) and (3), the contribution $v_{2,1}$ from the pole at $\alpha = K_{1N}$ can be written as

$$v_{2,1}(x, z) = -A \exp\{-iK_{1N}(x - x0\}\cos\{\sigma_{2N}(Z + H)\} \qquad \ldots(16)$$

which cancels exactly the incident Love wave as is to be expected. Let K_{2m} $(m = 1, 2, 3\ldots)$ denote the zeros of $I(\alpha, h_1)$ in the upper half plane. Then using

$$\sigma_{1m} = (K_{2m}^2 - k_1^2)^{1/2}, \quad \sigma_{2m} = (k_2^2 - K_{2m}^2)^{1/2}$$

we can write

$$v_{2,2} = 2\pi i \sum_{m=1}^{\infty} \frac{\cos\sigma_{2m}(Z+H)\, K_{2m}}{\sigma_{2m} h_1 (\sin\sigma_{2m} h_1)} \cos(\sigma_{2m} h_1)\left(\frac{C_{2m}}{U_{2m}} - 1\right)\exp(-iK_{2m}x)$$

$$\left[\frac{K_{2m} + k_2}{K_{2m} - k_2}\right]^{1/2} \frac{H_+(K_{2m})}{T_+(K_{2m})}\left[N_+(K_{2m}) - \frac{iA}{2\pi}\frac{\sigma_{2N}\exp(iK_{1N}\, x0)}{(K_{2m} - K_{1N})}\right.$$

$$\left.\times \left\{\frac{T_+(K_{1N})\sin\sigma_{2N}\,\delta_1}{H_+(K_{1N})(K_{1N} + k_2)} - \sin\sigma_{2N}\,\delta_2\,[P_+(K_{2m}) - P_+(K_{1N})]\right\}\right]. \qquad \ldots(17)$$

In (17), we have used

$$\left.\frac{dI(\alpha, h_1)}{d\alpha}\right|_{\xi = K_{2m}} = \frac{\sigma_{2m}^2 h_1}{K_{2m}\cos(\sigma_{2m} h_1)\left(\frac{C_{2m}}{U_{2m}} - 1\right)} \qquad \ldots(18)$$

where $c_{2m} = \frac{\omega}{K_{2m}}$ is the phase velocity of the Love type waves of the mth mode. It may be noted that (17) requires the relations $I(K_2 m) = 0$ which is equivalent to

$$\tan\sigma_{2m} h_1 = \nu \frac{\sigma_{1m}}{\sigma_{2m}}. \qquad \ldots(19)$$

Since (19) is the relation for the propagation Love type waves in layered structure consisting of a semi-infinite solid of rigidity μ_1 covered by a surface layer of uniform

thickness h_1 and rigidity μ_2, $v_{2,2}$ is therefore a Love propagating in the geometry as show in Fig. 1.

(*b*) *The region* $-h < z < -h; z < 0$

The transmitted wave $v_3(x, z)$ in this region is determined by applying inversion formula to eqn. (11). The contour of integration is closed in the upper half plane. The integrand has simple poles at $\alpha = K_{1N}$ and at the zeros of sinh $\sigma_2 h$ lying in the upper half plane. The contribution $v_{3,1}(x, z)$ arising from the pole at $\alpha = K_{1N}$ cancels the incident wave v_2^{inc} in this region. The contribution from the poles at $\alpha = ip_n$, where $P_n = \left[\frac{n^2\pi^2}{h^2} - k^2\right]^{1/2}$, denoted by $v_{3,2}$ is given by

$$v_{3,2} = \sqrt{2\pi i}\sum_{n=0}^{\infty} \exp(p_n x)\left[\frac{ip_n + k_2}{ip_n - k_2}\right]^{1/2} H_+(ip_n)\left[\frac{{}^*\cos\frac{\pi n}{h}(Z+H)}{T_+(ip_n)}\right.$$

$$- \frac{iA}{2\pi}\frac{\sigma_{2N}\exp(iK_{1N}x0)}{ip_n - K_{1N}}\left\{\frac{T_+(K_{1N})\sin(\sigma_{2N}\delta_1)}{(K_{1N}+k_2)H_+(K_{1N})}\right.$$

$$\left. - \sin\sigma_{2N}\delta_2[(P_+(ip_n) - P_+(K_{1N})]\right\}$$

$$- \frac{\cos\frac{n\pi}{h}(Z+h_1)}{Y_+(ip_n)}\left[O_+ - \frac{iA}{\sqrt{2\pi}}\frac{\sigma_{2N}\exp(iK_{1N}x0)}{(i_{pn} - K_{1N})}\right.$$

$$\left.\left.\times\left\{\frac{Y_+(K_{1N})\sin(\sigma_{2N}\delta_2)}{(K_{1N}+k_2)H_+(K_{1N})} - \sin(\sigma_{2N}\delta_1)(W_+(iP_n) - W_+(K_{1N}))\right\}\right]\right]^* \quad \ldots(20)$$

Note that in deriving eqn. (20), he have used the relation sinh $\sigma_2 h = 0$, which is the dispersion relation for waves in an infinite strip of uniform thickness δ and rigidity μ_2 with weak upper and lower surfaces at $z = -h_2$ and $z = -h_1$ respectively.

(*c*) *The region* $-H < z < -h_2; x < 0$

To find the transmitted waves in this range, we apply inversion formula to eqn. (12). Closing the contour of integration in the upper half plane, we find that the integrand has simple poles at $\alpha = K_{IN}$ and at zeros of sinh $\sigma_2\delta_2$ that lie in the upper half plane. The contribution, $v_{4,1}$, from $\alpha = K_{IN}$ exactly cancels the incident wave v_2^{inc} in this region. The poles arising from the zeros of sinh $\sigma_2\delta_2$ are $\alpha = ip_n' = \left\{k_2^2 - \frac{n^2\pi^2}{\delta^2}\right\}^{1/2}$, $n = 1, 2, 3, \ldots$ These poles give rise to the contribution

$$V_{4,2} = \sqrt{2\pi i}\sum_{n=0}^{\infty}\exp(p_n' x)\cos\frac{n\pi}{\delta_2}(Z+H)\left[\frac{ip_n' + k_2}{ip_n' - k_2}\right]^{1/2}\frac{H_+(ip_n')}{Y_+(ip_n')}$$

(*equation continued on p.* 593)

$$\times \left[Q_+ (ip'_n) - \frac{iA}{\sqrt{2\pi}} \frac{\sigma_{2N} \exp(iK_{IN} x_0)}{(ip'_n - K_{1N})} \left\{ \frac{Y_+ (K_{1N}) \sin \sigma_{2N} \delta_2}{(K_{1N}+k_2)\, H_+ (K_{1N})} \right.\right.$$

$$\left.\left. - \sin \sigma_{2N} \delta_1 (W_+ (ip'_n)\, W'_+ (K_{1N})) \right\} \right]. \quad ...(21)$$

The dispersion equation satisfied by $\nu_{4,2}$ corresponds to the relation $\sinh \sigma_2 \delta_2 = 0$, that is $\alpha = \left[k_2^2 - \frac{n^2 \pi^2}{\delta^2} \right]^{1/2}$, which is the dispersion relation for waves in an infinite strip of uniform thickness δ_2 and rigidity μ_2 with free upper surface and free/weak lower surface respectively.

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REFERENCES

1. S. Asghar, and F.D. Zaman, *Bull. Seismol. Soc. Am.* 76 (1986).
2. de Hoop A. T., Representation theorems for the displacement in an elastic solid and their application to elastodynamic diffraction theory, D. Sc. Thesis, Technische Hogeschool Delft, 1958.
3. D. S. Jones, *Q. Jl Math.* (2) 3 (1952), 189-96.
4. M. H. Kazi, *Bull. Seismol. Soc. Am.* 65 (1975), 146-81.
5. E. R. Lapwood, *Phil. Trans. R. Soc.* Lond. A 242 (1949), 63-100.
6. B. Noble, *Methods Based on the Wiener-Hopf Technique for the Solution of Partial Differential Equations*, Pergamon, New York, 1958.
7. R. Sato, *J. Phys. Earth* 9 (1961), 19-36.

APPENDIX A

The factorization of the function, involved in eqn. (8) and (9), has been fully described by Sato[7]. We only quote the results here.

(a) Let us write

$$\frac{\sinh \sigma_2 \delta_2}{\sigma_2 \delta} = \prod_{n=1}^{\infty} \left\{ p_n^2\, \bar{\delta}_n^2 + \alpha^2\, \bar{\delta}_n^2 \right\} = \bar{H}(\alpha) \quad ...(A.1)$$

where

$$p_n\, \bar{\delta}_n = \left[1 - k_2^2\, \bar{\delta}_n^2 \right]^{1/2} - i\,(k_2^2\, \bar{\delta}_n^2 - 2),\ \ \bar{\delta}_n = \frac{\delta_n}{\pi n}. \quad ...(A.2)$$

Also $\bar{H}(\alpha) = \bar{H}_+ \,.\, \bar{H}_-(\alpha)$, where

$$H_{\pm}(\alpha) = \prod_{n=1}^{\infty} \bar{p}_n \, \bar{\delta}_n \mp i\alpha \; \bar{\delta}_n + \exp\{\mp i\alpha \; \bar{\delta} + \bar{\chi}(\alpha)\}. \qquad \ldots(\text{A.3})$$

If

$$\bar{\chi}(\alpha) = - i\alpha \; \frac{\delta}{\pi}\left[1 - c - \log \frac{\alpha\delta}{\pi} + \frac{\alpha\delta}{2}\right], \text{ then } \; \bar{H}_{\pm}(\alpha) \sim |\, \alpha \,|^{-1/2}$$

as $|\, \alpha \,| \to \infty$

in appropriate half planes. Hence

$$\frac{\sinh \sigma_2 \, \delta}{\sigma_2 \, \delta} = \bar{H}_{+}(\alpha) \, . \, \bar{H}_{-}(\alpha). \qquad \ldots(\text{A.4})$$

(b) If $\pm \hat{K}_{1m}$ and $\pm \hat{K}_{2m}$ $(m = 1, 2, \ldots)$ denote the zeros of $I(\alpha, h_2)$ and $I(\alpha, h_1)$ respectively, we can write

$$\frac{I(\alpha, h_3)}{I(\alpha, h_1)} = \prod_{m=1}^{\infty} \left(\frac{\alpha^3 - \hat{K}^2_{1m}}{2 - K^2_{2m}}\right) \frac{G_1(\alpha)}{G_2(\alpha)}$$

where

$$G_1(\alpha) = I(\alpha, h_2) / \prod_{m=1} (\alpha^2 - \hat{K}^2_{1m})$$

$$G_2(\alpha) = I(\alpha, h_1) / \prod_{m=1}^{\infty} (\alpha^2 - \hat{K}^2_{2m})$$

and $G_{1,2}(\alpha)$ has no zeros.

Let

$$Q(\alpha) = \frac{G_1(\alpha)}{G_2(\alpha)} = Q_{+}(\alpha) \, . \, Q_{-}(\alpha)$$

then

$$\log Q_{+}(\alpha) = \frac{1}{2\pi i} \int_{\Gamma} \log \frac{Q_{+}(\omega)}{(\omega)\,\alpha} \, d\omega; \; Q_{-}(\alpha) = Q_{+}(-\alpha),$$

$$\frac{I(\alpha, h_2)}{I(\alpha, h_1)} = \prod_{m=1}^{\infty} \frac{(\alpha^2 - \hat{K}^2_{1m})}{(\alpha^2 - \hat{K}^2_{2m})} Q_{+}(\alpha) \, . \, Q_{-}(\alpha) = T_{+}(\alpha) \, T_{-}(\alpha) \qquad \ldots(\text{A.5})$$

where

$$T_{\pm}(\alpha) = Q_{\pm}(\alpha) \prod_{m=1}^{\infty} \left(\frac{\alpha + \hat{K}_{1m}}{\alpha + \hat{K}_{2m}}\right). \qquad \ldots(\text{A.6})$$

(c) The parallel calculation from (A1) to (A4) for $\frac{\sinh \sigma_2 \delta_1}{\sigma_2 \delta_1}$ and $\frac{\sinh \sigma_2 \delta_2}{\alpha_2 \delta_2}$ lead to

$$\frac{\sinh \alpha_2 \delta_1}{\sigma_2 \delta_1} \cdot \frac{\sigma_2 \delta_2}{\sinh \sigma_2 \delta_2} = Y_+(\alpha) \,.\, Y_-(\alpha) \qquad \ldots(A.7)$$

where

$$\frac{\sinh \sigma_2 \delta_1}{\sigma_2 \delta_1} = Y_{1+}(\alpha) \,.\, Y_{1-}(\alpha)$$

$$\frac{\sinh \sigma_2 \delta_2}{\sigma_2 \delta_2} = Y_{2+}(\alpha) \,.\, Y_{2-}(\alpha)$$

and

$$Y_\pm(\alpha) = \frac{Y_1 \pm (\alpha)}{Y_2 \pm (\alpha)}.$$

Appendix B

Following the general decomposition theorem (Noble[6]), the explicit respresentations $N_\pm(\alpha)$, $P_\pm(\alpha)$, $O_\pm(\alpha)$ and $W_\pm(\alpha)$ are given by

$$N_\pm(\alpha) = \pm \frac{1}{2\pi i} \int_{\substack{ic-\infty \\ id-\infty}}^{\substack{id+\infty \\ ic+\infty}} \frac{V'_{4+}(\xi, -h_2)\, d\xi}{(\xi + k_2)\, H_+(\xi)\, T_-(\xi)\, (\xi - \alpha)} \qquad \ldots(B.1)$$

$$P_\pm(\alpha) = \pm \frac{1}{2\pi i} \int \frac{d\xi}{(\xi + k_2)\, H_+(\xi) T_-(\xi)\, (\xi - \alpha)} \qquad \ldots(B.2)$$

$$O_\pm(\alpha) = \pm \frac{1}{2\pi i} \int \frac{V'_{2+}(\xi, -h)\, d\xi}{(\xi + k_2)\, H_+(\xi)\, Y_-(\xi)\, (\xi - \alpha)} \qquad \ldots(B.3)$$

$$W_\pm(\alpha) = \pm \frac{1}{2\pi i} \int \frac{d\xi}{(\xi + k_2)\, H_+(\xi)\, Y_-(\xi)\, (\xi - \alpha)} \qquad \ldots(B.4)$$

where $-\mathrm{Im}(k_1) < c < \mathrm{Im}(\alpha) < d < \mathrm{Im}(k_1)$.

The integrals in (B1), (B2), (B3) and (B4) can be calculated by the contour integration method. Let us consider

$$N_+(\alpha) = \frac{1}{2\pi i} \int_{ic-\infty}^{ic+\infty} \frac{V'_{4+}(\xi)\, d\xi}{(\xi + k_2)\, H_+(\xi)\, (\xi - \alpha)\, T_-(\xi)} \qquad \ldots(B.5)$$

For α lying in $(-\eta, \eta)$ we can write $\alpha = -\eta \cos \theta$, where $|\theta| < \pi$. Thus, eqn. (B5) can be written as

$$N_+(-\eta \cos\theta) = \frac{1}{2\pi i} \int_{ic-\infty}^{ic+\infty} \frac{V'_{4+}(\xi_1 - h_2) \prod_{m=1}^{\infty} (\xi - \hat{K}_{2m})\, d\xi}{(\xi+k_2)\, H_+(\xi)\, T_-(\xi)\, (\xi+\eta \cos\theta)} \qquad \ldots(B.6)$$

where $T_-(\xi) = Q_-(\xi) \prod_{m=1}^{\infty} \left[\frac{\xi - \hat{K}_{1m}}{\xi - \hat{K}_{2m}}\right]$, and $Q_-(\xi)$ has no zeros. Putting the value of $T_-(\xi)$ in (B6), we obtain

$$N_+(-\eta \cos\theta) = \frac{1}{2\pi i} \int_{ic-\infty}^{ic+\infty} \frac{V'_{4+}(\xi_1 - h_2) \prod_{m=1} (\xi - \hat{K}_{2m})\, d\xi}{(\xi+k_2)\, H_+(\xi)\, Q_-(\xi) \prod_{m=1}^{\infty} (\xi - \hat{K}_{1m})\, (\xi+\eta \cos\theta)}.$$

$$\ldots(B.7)$$

Closing the line of integration by a semi-circle in the upper half plane the poles captured are $\xi = -\eta \cos\theta$, $\xi = \hat{K}_{1m}$ $(m = 1, 2 \ldots)$. Thus

$$N_+(-\eta \cos\theta) = \sum_{m=1}^{\infty} \frac{V'_{4+}(K_{1l} - h_2)}{(\hat{K}_{1l} + k_2)\, H_+(\hat{K}_{1l})\, Q_+(\hat{K}_{1l})} \frac{\prod_{m=1}^{\infty} (\hat{K}_{1l} - \hat{K}_{2m})}{\prod_{m=1}^{\infty} (\hat{K}_{1l} - \hat{K}_{1m})}$$

$$+ \frac{V'_{4+}(-\eta \cos\theta, -h_2)}{(K_2 - \eta \cos\theta)\, H_+(-\eta \cos\theta)\, T_-(-\eta \cos\theta)}. \qquad \ldots(B.8)$$

The other integrals can be evaluated similarly.

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THE STUDY OF STREAMLINES OF M.G.D. FLOW OF A SURFACE S IN THE IMAGE SURFACE $\bar{S}$

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Let S be a surface and $\bar{S}$ be the spherical image or inverse surface or parallel surface of S. By knowing the relations of the first and second fundamental magnitudes of S and $\bar{S}$, we study the properties of streamlines of S in $\bar{S}$.

1. PRELIMINARIES

For steady flow the system of equations are given by[6]

$$\begin{aligned}
&\text{(i)}\ \nabla \cdot (\rho V) = 0\\
&\text{(ii)}\ \nabla \times (V \times H) = \nabla \times (\nu_H (\nabla \times H))\\
&\text{(iii)}\ \rho (V \cdot \nabla) V = \mu_e (H \cdot \nabla) H - \nabla\left(p + \mu_e \frac{H^2}{2}\right) + \nabla . T\\
&\text{(iv)}\ \rho (V \cdot \nabla h) = \nabla \cdot (V \cdot T) + \nabla \cdot (\kappa \nabla T) + (\nabla \times H)\\
&\qquad \times [\mu_e \nu_H (\nabla \times H) - \mu_e V \times H]\\
&\text{(v)}\ H = \alpha V \qquad \ldots(1.1)
\end{aligned}$$

where

ρ = density,

H = magnetic field vector,

V = velocity,

ν_H = magnetic viscosity,

p = pressure

α = scalar function.

T = stress tensor,

T = Temperature,

μ_e = magnetic permeability,

h = stagnation enthalpy,

κ = heat flux,

Using (1.1) (iii) in (1.1) (v) and ultimately using this in (1.1) (iv) taking account of the vector identity

$$(a \cdot \nabla) a = \text{curl } a \times a + \text{grad } \left(\frac{a^2}{2} \right)$$

and (1.1) (v) we get

$$\rho\,(V\cdot\nabla h)=V\cdot\left[\rho\left(\text{curl }V\times V+\text{grad}\left(\frac{V^2}{2}\right)\right)-\mu_e\left(\text{curl }(\alpha V)\times\alpha V+\text{grad}\left(\frac{\alpha^2V^2}{2}\right)\right)+\nabla\left(p+\mu_e\frac{\alpha^2V^2}{2}\right)\right].\,T_{ij}\,e_{ij}$$
$$+\nabla\cdot(\kappa\nabla\dot{T})+(\nabla\times H)\,(\mu_e\,\nu_H\,(\nabla\times H)-\mu_e\,V\times H). \qquad \ldots(1.2)$$

Using (1.1) (v) we have

$$V\times H=0 \qquad \ldots(1.3)$$

$$\text{curl }(\alpha V)\times(\alpha V)=\alpha^2\,\text{curl }V\times V \qquad \ldots(1.4)$$

For two dimensional flow

$$\Sigma\,T_{ij}\,e_{ij}=T_{11}\,e_{11}+T_{12}\,e_{12}+T_{21}\,e_{21}+T_{22}\,e_{22}\,\cdot \qquad \ldots(1.5)$$

Using (1.3), (1.4) and (1.5) in (1.2) and properties of vector analysis we get as

$$\rho\,(V\cdot\nabla h)=\rho\,V\cdot\text{grad}\left(\frac{V^2}{2}\right)-\mu_e\,V\cdot\text{grad}\left(\frac{\alpha^2V^2}{2}\right)$$
$$+V\cdot\nabla\left(p+\mu_e\frac{\alpha^2V^2}{2}\right)+\nabla\cdot(\kappa\nabla T)$$
$$+\mu_e\,\nu_H\,(\nabla\times H)^2+T_{11}\,e_{11}+T_{12}\,e_{12}+T_{22}\,e_{22}+T_{21}\,e_{21}. \qquad \ldots(1.6)$$

Let

$$V=V(\xi,\eta)\,e_l. \qquad \ldots(1.7)$$

In natural i.e. streamline co-ordinates with $g_1(\xi,\eta)\,d\,\xi$ and $g_2(\xi,\eta)\,d\eta$ as the components of a vector element of are length we have the following system of equations (1.8)

$$\text{(i) div }(\rho V)=\frac{1}{g_1\,g_2}\frac{\partial}{\partial\xi}(g_2\,\rho V)$$

$$\text{(ii) grad }\frac{V^2}{2}=\frac{1}{g_1}\frac{\partial}{\partial\xi}\left(\frac{V^2}{2}\right)e_1+\frac{1}{g_2}\frac{\partial}{\partial\eta}\left(\frac{V^2}{2}\right)e_2$$

$$\text{(iii) grad}\left(\frac{\alpha^2V^2}{2}\right)=\frac{1}{g_1}\frac{\partial}{\partial\xi}\left(\frac{\alpha^2V^2}{2}\right)e_1+\frac{1}{g_2}\frac{\partial}{\partial\eta}\left(\frac{\alpha^2V^2}{2}\right)e_2$$

$$\text{(iv) grad }h=\frac{1}{g_1}\frac{\partial h}{\partial\xi}\,e_1+\frac{1}{g_2}\frac{\partial h}{\partial\xi}\,e_2$$

$$\text{(v) grad}\left(p+\mu_e\frac{\alpha^2V^2}{2}\right)=\frac{1}{g_1}\frac{\partial}{\partial\xi}\left(p+\mu_e\frac{\alpha^2V^2}{2}\right)e_1+\frac{1}{g_2}\frac{\partial}{\partial\eta}\times\left(p+\mu_e\frac{\alpha^2V^2}{2}\right)e_2$$

$$\text{(vi) grad } \dot{T} = \frac{1}{g_1}\frac{\partial T}{\partial \xi} e_1 + \frac{1}{g_2}\frac{\partial T}{\partial \eta} e_2$$

$$\kappa \text{ grad } \dot{T} = \frac{\kappa}{g_1}\frac{\partial \dot{T}}{\partial \xi} e_1 + \frac{\kappa}{g_2}\frac{\partial \dot{T}}{\partial \eta} e_2$$

$$\text{(vii) div } (\kappa \text{ grad } \dot{T}) = \frac{1}{g_1 g_2}\left[\frac{\partial}{\partial \xi}\left(\frac{\kappa}{g_1}\frac{\partial \dot{T}}{\partial \xi} g_2\right) + \frac{\partial}{\partial \eta}\left(\frac{\kappa}{g_2}\frac{\partial \dot{T}}{\partial \eta} g_1\right)\right]$$

$$\text{(viii) curl } H = \nabla \times H = \nabla \times (\alpha V) = \text{grad } \alpha \times V + \alpha \text{ curl } V$$

$$\text{grad } \alpha = \frac{1}{g_1}\frac{\partial \alpha}{\partial \xi} e_1 + \frac{1}{g_2}\frac{\partial \alpha}{\partial \eta} e_2$$

$$\text{grad } \alpha \times V = -\frac{V}{g_2}\frac{\partial \alpha}{\partial \eta} e_3$$

$$\text{curl } V = -\frac{1}{g_1 g_2}\frac{\partial}{\partial \eta}(g_1 V)\, e_3$$

$$\nabla \times H = -\left[\frac{V}{g_2}\frac{\partial \alpha}{\partial \eta} + \frac{\alpha}{g_1 g_2}\frac{\partial}{\partial \eta}(g_1 V)\right]$$

$$e_3 = -\frac{1}{g_1 g_2}\frac{\partial}{\partial \eta}(g_1 \alpha V)\, e_3$$

$$\text{(ix) } (\nabla \times H)^2 = \frac{1}{g_1^2 g_2^2}\left(\frac{\partial}{\partial \eta}\left(g_1 \alpha V\right)\right)^2. \qquad \dots(1.8)$$

Using (1.7) and the properties of stress tensor in (1.5) we have

$$e_{11} = \frac{2}{g_1}\frac{\partial V}{\partial \xi},\; e_{12} = \frac{g_1}{g_2}\frac{\partial}{\partial \eta}\left(\frac{V}{g_1}\right) \text{ and } e_{13} = \frac{2V}{g_1 g_2}\frac{\partial g_2}{\partial \xi}$$

and

$$\sum_{j=1}^{2} T_{ij} e_{ij} = \frac{2}{g_1}\left(-p + \frac{2\nu_H}{g_1}\frac{\partial V}{\partial \xi}\right)\frac{\partial V}{\partial \xi} + 2\nu_H \frac{g_1^2}{g_2^2}\left(\frac{\partial}{\partial \eta}\left(\frac{V}{g_1}\right)\right)^2$$

$$+ \frac{2V}{g_1 g_2}\left(-p + \frac{2\nu_H V}{g_1 g_2}\frac{\partial g_2}{\partial \xi}\right)\frac{\partial g_2}{\partial \xi} \qquad \dots(1.9)$$

Here V is the magnitude of velocity, e_1 a unit vector in the direction to the velocity, e_2 a unit vector in the direction perpendicular to the velocity but in the plane of flow, e_3 a unit vector in the direction of normal to the plane of flow such that e_1, e_2 and e_3 from a right handed system. The metric of this (ξ, η) net is of the form

$$ds^2 = g_1^2(\xi, \eta)\, d\xi^2 + g_2^2(\xi, \eta)\, d\eta^2$$

where g_1 and g_2 satisfy the Gauss equation

$$\frac{\partial}{\partial \xi}\left(\frac{1}{g_1}\frac{\partial g_2}{\partial \xi}\right)+\frac{\partial}{\partial \eta}\left(\frac{1}{g_2}\frac{\partial g_1}{\partial \eta}\right)=0.$$

Using (1.8) (ii) to (x) in (1.6) we have

$$\begin{aligned}\frac{\rho V}{g_1}\frac{\partial h}{\partial \xi} &= \frac{1}{g_1 g_2}\left[\frac{\partial}{\partial \xi}\left(\frac{\kappa}{g_1}\frac{\partial T}{\partial \xi}g_2\right)+\frac{\partial}{\partial \eta}\left(\frac{\kappa}{g_2}\frac{\partial T}{\partial \eta}g_1\right)\right]\\ &+\frac{\mu_e \nu_H}{g_1^2 g_2^2}\left(\frac{\partial}{\partial \eta}\left(g_1 \alpha V\right)\right)^2+\frac{\rho V}{g_1}\frac{\partial}{\partial \xi}\left(\frac{V^2}{2}\right)\\ &-\mu_e\frac{V}{g_1}\frac{\partial}{\partial \xi}\left(\frac{\alpha^2 V^2}{2}\right)+\frac{2}{g_1}\left(-p+\frac{2\nu_H}{g_1}\frac{\partial V}{\partial \xi}\right)\frac{\partial V}{\partial \xi}\\ &+2\nu_H\frac{g_1^2}{g_2^2}\left(\frac{\partial}{\partial \eta}\left(\frac{V}{g_1}\right)\right)^2+\frac{2V}{g_1 g_2}\\ &\times\left(-p+\frac{2\nu_H}{g_1 g_2}\frac{\partial g_2}{\partial \xi}\right)\frac{\partial g_2}{\partial \xi}. \qquad \ldots(1.10)\end{aligned}$$

From (1.1) (i) and (1.8) (i) we have

$$\frac{\partial}{\partial \xi}(g_2 \rho V)=0 \qquad \ldots(1.11)$$

and this implies

$$g_2 \rho V = f(\eta)$$

implies

$$V=\frac{f(\eta)}{g_2\rho}.$$

Using (1.11) in (1.10) and cancelling the term $1/g_1 g_2$ we have

$$\begin{aligned}f_2(\eta)\frac{\partial h}{\partial \xi} &= \frac{\partial}{\partial \xi}\left(\frac{\kappa}{g_1}\frac{\partial T}{\partial \xi}g_2\right)+\frac{\partial}{\partial \eta}\left(\frac{\kappa}{g_2}\frac{\partial T}{\partial \eta}g_1\right)\\ &+\frac{\mu_e \nu_H}{g_1 g_2}\left(\frac{\partial}{\partial \eta}\left(\frac{g_1 \alpha f(\eta)}{g_2\rho}\right)\right)^2+\frac{f(\eta)}{2}\frac{\partial}{\partial \xi}\left(\frac{f^2(\eta)}{g_2^2\rho^2}\right)\\ &-\frac{\mu_e f(\eta)}{2\rho}\frac{\partial}{\partial \xi}\left(\frac{\alpha^2 f^2(\eta)}{g_2^2\rho^2}\right)+\frac{2\nu_H g_1^3}{g_2}\frac{\partial}{\partial \eta}\left(\frac{f(\eta)}{g_2 g_1}\right)\\ &+\frac{2f(\eta)}{g_2\rho}\left(-p+\frac{2\nu_H f(\eta)}{g_1^2 g_2^2}\frac{\partial g_2}{\partial \xi}\right)\frac{\partial g_2}{\partial \xi}\\ &+\frac{2g_2}{1}\left(-p+\frac{2\nu_H}{g_1}\frac{\partial}{\partial \xi}\left(\frac{f(\eta)}{g_2\partial}\right)\right)\frac{\partial}{\partial \xi}\left(\frac{f(\eta)}{g_2\rho}\right). \qquad \ldots(1.12)\end{aligned}$$

We prescribe the streamlines to be straight lines. We assume that they are not parallel but envelope a curve C. We now take the tangents to the curve and their orthogonal tracjectories the involute of c as a sytem of curvilinear co-ordinates. The squared element of arc length ds in this or-thogonal curvilinear co-ordinate system is given by

$$ds^2 = d\xi^2 + (\xi - \sigma(\eta))^2 \, d\eta^2 \qquad \ldots(1.13)$$

where $\sigma = \sigma(\eta)$ denotes the arc length of the curve and η the angle subtended by the tangent line with x-axis and ξ is a parameter constant along the involute of this co-ordinate system, the co-ordinate curves ξ = constant are the orthogonal trajectories (streamlines) are the involutes of the curve C and η = constant (the streamlines) are its tangent lines.

Hence by[7]

$$g_1(\xi, n) = 1, \; g_2 = \xi - \sigma(\eta). \qquad \ldots(1.14)$$

2. THE STUDY OF STREAMLINES OF MGD FLOW OF A SURFACE S IN THE SPHERICAL IMAGE $\bar{S}$

The study of spherical image is a spherical representation of a point, differential cross-section or partial configuration of the surface on a unit sphere whose centre may be considered as origin. The movement of a point on the curve on a given surface has a corresponding relational movement on the sphere. It can be mentioned in this connection that one of the recent developments in nuclear magnetic resonance spectroscopy is to consider the spectrum by a surface in 3-dimensional space in constrast to the higher to hold notion that it is a graph in terms of absorption intensity as a function of frequency. This is one case where in the concept of spherical representation can be inculcated to study the spherically nuclear magnetic resonance spectral properties.

Let $\bar{S}$ be the spherical image of a surface S. Let n be the unit normal at the point P on the surface S, the point Q whose position vector is n is said to correspond to P or to be the image of P. Clearly Q lies on the unit sphere $\bar{S}$. The position vector R of Q on $\bar{S}$ is given by

$$R = n. \qquad \ldots(2.1)$$

Differentiating (2.1) w.r.t. ξ and η we have

$$R_1 = n_1 = W^{-2}\,[(FM - GL)\,r_1 + (FL - EM)\,r_2] \qquad \ldots(2.2)$$

$$R_2 = n_2 = W^{-2}\,[(FN - GM)\,r_1 + (FM - EN)\,r_2], \; R_1 = \frac{\partial R}{\partial \xi},$$

$$R_2 = \frac{\partial R}{\partial \eta} \qquad \ldots(2.3)$$

where r is the position vector of P and E, F, G and L, M, N are the first and second fundamental magnitudes on the surface S and (ξ, η) is a curvilinear co-ordinate system on the surface S.

Let e, f, g denote the first fundamental magnitudes for the spherical image $\bar{S}$. They are defined by

$$e = R_1 \cdot R_1, f = R_1 \cdot R_2 \text{ and } g = R_2 \cdot R_2. \qquad \ldots(2.4)$$

Using (2.2), (2.3) in (2.4) we have

$$e = W^{-2} (EM^2 - 2FLM + GL^2) \qquad \ldots(2.5)$$

$$f = W^{-2} (EMN - FM^2 - FLM + GLM) \qquad \ldots(2.6)$$

$$g = W^{-2} (EN^2 - 2FMN + GM^2). \qquad \ldots(2.7)$$

Let the streamlines in the surface S be straight lines. Using (1.14) in (2.5), (2.6) and (2.7) we have

$$e = (\xi - \sigma(\eta))^{-2} [M^2 + (\xi - \sigma(\eta))^2 L^2] \qquad \ldots(2.8)$$

$$f = (\xi - \sigma(\eta))^{-2} [MN + (\xi - \sigma(\eta))^2 LM] \qquad \ldots(2.9)$$

$$g = (\xi - \sigma(\eta))^2 [N^2 + (\xi - \sigma(\eta))^2 M^2]. \qquad \ldots(2.10)$$

The image curves of ξ = constant and η = constant of S will be orthogonal in $\bar{S}$ provided $f = 0$ i.e. $M = 0$. Hence the image curves of ξ = constant and η= constant will be orthogonal in $\bar{S}$ if ξ and η are lines of curvature because $F = 0$ and $M = 0$ is the necessary and suficient condition for the curves ξ = constant and η = constant to be lines of curvature. In this case if k_1 and k_2 are the principal curvatures, these are given by

$$L = k_1 E, \; N = k_2 G \qquad \ldots(2.11)$$

where k_1 and k_2 are functions of ξ, η. Hence using (1.14) we have

$$L = k_1, \; M = 0, \; N = k_2^2 (\xi - \sigma(\eta))^2. \qquad \ldots(2.12)$$

Using (2.12) in (2.8), (2.9) and (2.10) we have

$$e = k_1^2, \; f = 0, \; g = k_2^2 (\xi - \sigma(\eta)). \qquad \ldots(2.13)$$

Remark 2.1 : If the lines of curvatures ξ = constant and η = constant are parametric curve then

$$\frac{\partial k_1}{\partial \eta} = \frac{1}{2} \frac{k_2 - k_1}{E} E_2 \text{ and } \frac{\partial k_2}{\partial \xi} = \frac{1}{2} \frac{k_1 - k_2}{G} G_1 \qquad \ldots(2.14)$$

where E_2 and G_1 denote the partial differentiation of E and G w.r.t. η and ξ respectively. We have seen that the flow in the surface S is planar.

Similarly we can show that the image of the flow of S in the spherical image $\bar{S}$ is planar by using (2.13) and (2.14) and

$$K = -\frac{1}{\sqrt{eg}}\left(\frac{\partial}{\partial\xi}\left(\frac{1}{\sqrt{e}}\frac{\partial\sqrt{g}}{\partial\xi}\right)+\frac{\partial}{\partial\xi}\left(\frac{1}{\sqrt{g}}\frac{\partial\sqrt{e}}{\partial\eta}\right)\right)=0.$$

Take $e = g_1^2(\xi,\eta) = k_1^2$ and $g = g_2^2(\xi,\eta) = k_2^2(\xi-\sigma(\eta))^2$ and those imply

$$g_1(\xi,\eta) = k_1,\ g_2(\xi,\eta) = k_2(\xi-\sigma(\eta))^2. \qquad \text{...(2.15)}$$

Theorem 2.1—If the streamlines in MGD in the surface S are straight lines but not parallel and envelope a curve C and the tangent lines to C i.e. η = constant and the involute of the curve C i.e. ξ = constant are taken as orthogonal curvilinear coordinate then the image streamlines of S in the sphere $\bar{S}$ are concentric circles.

PROOF: Using (2.15) in (1.12) we have

$$\begin{aligned}
f(\eta)\frac{\partial h}{\partial\xi} &= \frac{\partial}{\partial\xi}\left(\frac{\kappa}{k_1}k_2(\xi-\sigma(\eta))\frac{\partial T}{\partial\xi}\right) \\
&+\frac{\partial}{\partial\eta}\left(\frac{\kappa}{k_2(\xi-\sigma(\eta))}\frac{\partial T}{\partial\eta}k_1\right)+\frac{\mu_e\nu_H}{k_1k_2(\xi-\sigma(\eta))} \\
&\times\left[\frac{\partial}{\partial\eta}\left(\frac{k_1\alpha f(\eta)}{k_2(\xi-\sigma(\eta))^2}\right)\right]^2+\frac{f(\eta)}{2}\frac{\partial}{\partial\xi}\left(\frac{f^2(\eta)}{\rho^2k_2^2(\xi-\sigma(\eta))^2}\right) \\
&-\frac{\mu_e f(\eta)}{2\rho}\frac{\partial}{\partial\xi}\left(\frac{\alpha^2 f(\eta)}{\rho^2k_2^2(\xi-\sigma(\eta))^2}\right)+\frac{2\nu_H k_1^3}{k_2(\xi-\sigma(\eta))} \\
&\times\left[\frac{\partial}{\partial\eta}\left(\frac{f(\eta)}{\rho k_1k_2(\xi-\sigma(\eta))}\right)\right]^2+\frac{2f(\eta)}{\rho k_2(\xi-\sigma(\eta))} \\
&\times\left[-p+\frac{2\nu_H f(\eta)}{\rho k_1k_2^2(\xi-\sigma)^2}\right]\times\left(\frac{\partial k_2}{\partial\xi}(\xi-\sigma)+k_2\right) \\
&+2k_2(\xi-\sigma)\left[-p+\frac{2\nu_H}{k_1}\frac{\partial}{\partial\xi}\left(\frac{f(\eta)}{\rho k_2(\xi-\sigma)}\right)\right] \\
&\times\frac{\partial}{\partial\xi}\left(\frac{f(\eta)}{\rho k_2(\xi-\sigma)}\right).
\end{aligned}$$

Simplifying the above equation we get

$$\begin{aligned}
f(\eta)\frac{\partial h}{\partial\xi} &= \frac{\partial}{\partial\xi}\left(\frac{\kappa k_2}{k_1}\right)(\xi-\sigma)\frac{\partial T}{\partial\xi}+\frac{\kappa k_2}{k_1}\frac{\partial T}{\partial\xi}+\frac{\kappa k_2}{k_1}(\xi-\sigma)\frac{\partial^2 T}{\partial\xi^2} \\
&+\frac{\partial}{\partial\eta}\left(\frac{\kappa k_1}{k_2}\right)\frac{1}{(\xi-\sigma)}\frac{\partial T}{\partial\eta}+\frac{\kappa k_1}{k_2(\xi-\sigma)}\frac{\partial^2 T}{\partial\eta^2}
\end{aligned}$$

(equation continued on p. 604)

$$+ \frac{\kappa\, \sigma'(\eta)\, k_1}{k_2 (\xi - \sigma)^2} \frac{\partial T}{\partial \eta} + \frac{\mu_e\, \nu_H}{k_1 k_2 (\xi - \sigma)} \Bigg[\frac{\partial}{\partial \eta} \left(\frac{k_1\, \alpha\, f(\eta)}{\rho\, k_2} \right) \frac{1}{\xi - \sigma}$$

$$+ \frac{k_1\, \sigma' f(\eta)}{k_2 (\xi - \sigma)^2} \Bigg]^2 + \frac{f^3(\eta)}{2} \Bigg[\frac{\partial}{\partial \xi} \left(\frac{1}{\rho^2 k_2^2} \right)$$

$$\left(\frac{1}{(\xi - \sigma)^2} - \frac{2}{\rho^2 k_2^2 (\xi - \sigma)^3} \right] - \frac{\mu_e f^3(\eta)}{2}$$

$$\times \left[\frac{\partial}{\partial \xi} \left(\frac{\alpha^2}{\rho^2 k_2^2} \right) \frac{1}{(\xi - \sigma)^2} - \frac{2}{\rho^2 k_2^2 (\xi - \sigma)^3} \right]$$

$$+ \frac{2\nu_H}{k_2 (\xi - \sigma(\eta))} \times \left[\frac{\partial}{\partial \eta} \left(\frac{f(\eta)}{\rho\, k_1 k_2} \right) \frac{1}{\xi - \sigma} + \frac{f(\eta)\, \sigma'(\eta)}{\rho k_1 k_2 (\xi - \sigma)^2} \right]^2$$

$$+ \frac{2 f(\eta)}{\rho k_2 (\xi - \sigma)} \times \Bigg[- p + \frac{2\nu_H f(\eta)}{\rho k_1 k_2^2 (\xi - \sigma)^2}$$

$$\times \left(\frac{\partial k_2}{\partial \xi} (\xi - \sigma) + k_2 \right) \Bigg] \times \left[\frac{\partial k_2}{\partial \xi} (\xi - \sigma) + k_2 \right]$$

$$+ 2 k_2 (\xi - \sigma) \Bigg[- p + \frac{2\nu_H}{k_1} f(\eta) \Bigg\{ \frac{\partial}{\partial \xi} \left(\frac{1}{\rho k_2} \right) \frac{1}{\xi - \sigma}$$

$$- \frac{1}{\rho k_2 (\xi - \sigma)^2} \Bigg\} \Bigg] f(\eta) \Bigg[\frac{\partial}{\partial \xi} \left(\frac{1}{\rho k_2} \right) \frac{1}{\xi - \sigma}$$

$$- \frac{1}{\rho k_2 (\xi - \sigma)^2} \Bigg].$$

Multiply the above equation by $(\xi - \sigma(\eta))^5$ we get

$$(\xi - \sigma)^5 f(\eta) \frac{\partial h}{\partial \xi} = (\xi - \sigma)^6 \frac{\partial}{\partial \xi} \left(\frac{\kappa\, k_2}{k_1} \right) \frac{\partial T}{\partial \xi} + \frac{\kappa\, k_2}{k_2} (\xi - \sigma)^5 \frac{\partial T}{\partial \xi}$$

$$+ \frac{\kappa\, k_2}{k_1} (\xi - \sigma)^6 \frac{\partial^2 T}{\partial \xi^2} + (\xi - \sigma)^4 \frac{\partial}{\partial \eta} \left(\frac{\kappa\, k_1}{k_2} \right) \frac{\partial T}{\partial \eta}$$

$$+ \frac{\kappa\, k_1 (\xi - \sigma)^4}{k_2} \frac{\partial^2 T}{\partial \eta^2} + \frac{\kappa\, \sigma'(\eta)\, k_1}{k_2} (\xi - \sigma)^3 \frac{\partial T}{\partial \eta}$$

$$+ \frac{\mu_e\, \nu_H}{k_1 k_2} \left[\frac{\partial}{\partial \eta} \left(\frac{k_1\, \alpha\, f(\eta)}{\rho k_2} \right) (\xi - \sigma) + \frac{k_1\, \alpha\, \sigma' f(\eta)}{\rho k_2} \right]^2$$

$$+ \frac{f^3(\eta)}{2} \left[\frac{\partial}{\partial \xi} \left(\frac{1}{\rho^2 k_2^2} \right) (\xi - \sigma)^3 - \frac{2}{\rho^2 k_2^2} (\xi - \sigma)^2 \right]$$

$$- \frac{\mu_e f^3(\eta)}{2\rho} \left[\frac{\partial}{\partial \xi} \left(\frac{\alpha^2}{\rho^2 k_2^2} \right) (\xi - \sigma)^3 - \frac{2\alpha^2}{\rho^2 k_2^2} (\xi - \sigma)^2 \right]$$

(equation continued on p. 605)

$$+ \frac{2\nu_H k_1}{k_2}\left[\frac{\partial}{\partial\eta}\left(\frac{f(\eta)}{\rho k_1 k_2}\right)(\xi - \sigma) + \frac{f(\eta)\,\sigma'(\eta)}{\rho k_1 k_2}\right]^2$$

$$+ \frac{2f(\eta)}{\rho k_2}\left[-p + \frac{2\nu_H f(\eta)}{\rho k_1 k_2^2}\left(\frac{\partial k_2}{\partial\xi}(\xi - \sigma) + k_2\right)\right]$$

$$\times\left[\frac{\partial k_2}{\partial\xi}(\xi - \sigma) + k_2(\xi - \sigma)^2\right] + 2k_2(\xi - \sigma)^2 f(\eta)$$

$$\times\left[-p(\xi - \sigma)^2 + \frac{2\nu_H f(\eta)}{k_1}\left(\frac{\partial}{\partial\xi}\left(\frac{1}{\rho k_2}\right)\right.\right.$$

$$\left.\left.\times(\xi - \sigma) - \frac{1}{\rho k_2}\right)\right] \times \left[(\xi - \sigma)\frac{\partial}{\partial\xi}\left(\frac{1}{\rho k_2}\right) - \frac{1}{\rho k_2}\right]. \qquad \ldots(2.16)$$

Equation (2.16) to hold identically we must have $\xi = \sigma(\eta)$. Using this in (2.16) we have

$$\frac{\mu_e \nu_H}{k_1 k_2}\frac{k_1^2 \alpha^2 (\sigma')^2 f^2(\eta)}{\rho^2 k_2^2} + \frac{2\nu_H k_1}{k_2}\frac{f^2(\eta)\,\sigma'^2(\eta)}{\rho^2 k_1^2 k_2^2} = 0. \qquad \ldots(2.17)$$

Simplifying (2.17) we have

$$f^2(\eta)\,\sigma'^2(\eta)\,\nu_H\left(\mu_e k_1 \alpha_2 + \frac{2}{k_1}\right) = 0$$

implies

$$f^2(\eta)\,\sigma'^2(\eta)\,\nu_H = 0.$$

Since $f(\eta) \neq 0$ and $\nu_H \neq 0$ we must have $\sigma'(\eta) = 0$.

This shows that streamlines are concentric circles.

3. The Study of Streamlines of MGD Flow of a Surface S in the Inverse Surface $\bar{S}$

As the name inverse surface itself is suggestive of is the surface corresponding to the given surface which has the same shape, size when viewed from a crorresponding point on the given surface. In other words concavity and convexity characteristic of a given surface assume the converse characteristics on an inverse surface. An analysis of inverse surface with that of the given surface helps us to study a surface at different orientations Gaseous flow, incompressible, compressible all encounter such sort of flow spots simultaneously and consecutively. It is true that a perfectly geometrically congruent isometric natured conformal charactered matched surface and inverse surface is difficult to come across in practice. An approximate evaluation of the pattern of flow can be made if such an assumption were made that a crest and crevice, a mountain and precipice are taken to be set of given surface and inverse surface. With the

establishment of the new fundamental magnitudes for the inverse surface and knowledge of intrinsic relationship between the fundamental magnitudes which determine surface and its inverse surface, it is possible to maintain the same tempo of flow.

Let $\bar{S}$ be the inverse surface of S. Let the center of inversion be taken as the origin. If c is the radius of inversion, the position vector $\bar{r}$ of a point on the inverse surface $\bar{S}$ corresponding to the point r on S has the direction of r and magnitude c^2/r^2. It is given by

$$\bar{r} = \frac{c^2}{r^2}\, \mathbf{r}. \qquad \ldots(3.1)$$

Differentiating (3.1) w.r.t. ξ and η we have

$$\bar{r}_1 = \frac{\partial \bar{r}}{\partial \xi} = \frac{c^2}{r^2}\, r_1 - \frac{2c^2}{r^2}\, r_1\, r \qquad \ldots(3.2)$$

$$\bar{r}_2 = \frac{\partial \bar{r}}{\partial \eta} = \frac{c^2}{r^2}\, r_2 - \frac{2c^2}{r^3}\, r_2\, r. \qquad \ldots(3.3)$$

Using (3.2), (3.3) the first fundamental magnitudes, $\bar{E}$, F, $\bar{G}$ of $\bar{S}$ are given by

$$\bar{E} = \bar{r}_1 \cdot \bar{r}_1 = \frac{c^4}{r^4} E, \qquad \bar{R} = \bar{r}_1 \cdot \bar{r}_2 = \frac{c^4}{r^4} F,$$

$$\bar{G} = \bar{r}_2 \cdot \bar{r}_2 = \frac{c^4}{r^4} G, \text{ and} \qquad \bar{W} = \frac{c^4}{r^4} W. \qquad \ldots(3.4)$$

By assuming the streamlines in the surface S to be streamlines we establish that the streamlines in the inverse surface $\bar{S}$ are also streamlines by using the relations of first and second fundamental magnitudes of S and $\bar{S}$.

Using (1.14) in (3.4) we have

$$\bar{E} = \frac{c^4}{r^4},\ \bar{F} = 0,\ \bar{G} = \frac{c^4}{c^4} (\xi - \sigma(\eta))^2 \text{ and } \bar{W}^2 = \frac{c^8}{r^8} (\xi - \sigma(\eta))^2. \qquad \ldots(3.5)$$

We have already seen that the flow in S is planar the Gauss equation corresponding to the fundamental magnitudes of $\bar{S}$ is given by

$$\bar{K} = \frac{-1}{\sqrt{\bar{E}\bar{G}}} \left(\frac{\partial}{\partial \xi} \left(\frac{1}{\sqrt{\bar{E}}} \frac{\partial \sqrt{\bar{G}}}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left(\frac{1}{\sqrt{\bar{G}}} \frac{\partial \sqrt{\bar{E}}}{\partial \eta} \right) \right). \qquad \ldots(3.6)$$

Take $g_1^2(\xi, \eta) = \bar{E} = \dfrac{c^4}{r^4}$ and $g_2^2(\xi, \eta) = \bar{G} = \dfrac{c^4}{r^4} (\xi - \sigma(\eta))^2$ ...(3.7)

implies

$$g_1 = \frac{c^2}{r^2} \text{ and } g_2 = \frac{c^2}{r^2} (\xi - \sigma).$$

Lemma 3.1—If flow in the surface S is planar then the image flow on the inverse surface $\bar{S}$ of S is either elliptic or hyperbolic.

PROOF : Using (3.5) in (3.6) we have

$$\bar{K} = -\frac{r^4}{c^4(\xi-\sigma(\eta))}\left[\frac{\partial}{\partial\xi}\left(\frac{r^2}{c^2}\frac{\partial}{\partial\xi}\left(\frac{c^2}{r^2}\left(\xi-\sigma(\eta)\right)\right)\right)\right.$$

$$\left.+\frac{\partial}{\partial\eta}\left(\frac{r^2}{c^2(\xi-\sigma(\eta))}\frac{\partial}{\partial\eta}\left(\frac{c^2}{r^2}\right)\right)\right]$$

$$\bar{K} = -\frac{r^4}{c^4(\xi-\sigma(\eta))}\left[\frac{\partial}{\partial\xi}\left(r^2\frac{\partial}{\partial\xi}\left(\frac{\xi-\sigma(\eta)}{r^2}\right)\right)\right.$$

$$\left.+\frac{\partial}{\partial\eta}\left(\frac{r^2}{\xi-\sigma(\eta)}\frac{\partial}{\partial\eta}\left(\frac{1}{r^2}\right)\right)\right] = -\frac{r^4}{c^4(\xi-\sigma(\eta))}$$

$$\times\left[\frac{\partial}{\partial\xi}\left\{r^2\left(\frac{1}{r^2}-\frac{2(\xi-\sigma(\eta))}{r^3}\right)r_1\right\}+\frac{\partial}{\partial\eta}\left(\frac{r^2}{\xi-\sigma(\eta)}\frac{-2r^2}{r^3}\right)\right]$$

$$-\frac{r^4}{c^4(\xi-\sigma(\eta))}\left[\frac{\partial}{\partial\xi}\left(1-\frac{2(\xi-\sigma(\eta))}{r}r_1\right)\right.$$

$$\left.-\frac{2\partial}{\partial\eta}\left(\frac{r_2}{r(\xi-\sigma(\eta))}\right)\right]$$

$$\bar{K} = -\frac{r^4}{c^4(\xi-\sigma(\eta))}\left[-\frac{2r_1}{r}\frac{2(\xi-\sigma(\eta))}{r}r_{11}+\frac{2(\xi-\sigma(\eta))}{r}r_1^2\right.$$

$$\left.-2\frac{r_{22}}{r(\xi-\sigma(\eta))}+\frac{2r_2^2}{r^2(\xi-\sigma(\eta))}-\frac{2r_2}{r(\xi-\sigma(\eta))^2}\right]$$

$$\bar{K} = \frac{2r^3}{c^4(\xi-\sigma(\eta))}\left[r_1+(\xi-\sigma(\eta))r_{11}-\frac{\xi-\sigma(\eta)}{r}r_1^2\right.$$

$$\left.+\frac{r_{22}}{\xi-\sigma(\eta)}+\frac{r_2}{(\xi-\sigma(\eta))^2}-\frac{r_2^2}{r(\xi-\sigma(\eta))}\right].$$

Thus $\bar{K}$ is either positive or negative. Hence the image flow in the inverse surface $\bar{S}$ of S is either elliptic or hyperbolic by

Theorem 3.1—If the streamlines in MGD in the surface S are straight lines but not parallel and envelope a curve C and the tangent lines to C i.e. η = constant and the involute of the curve C i.e. ξ = constant are taken as orthogonal curvilinear coordinates then the image streamlines of S in the inverse surface $\bar{S}$ of S are either concentric circles or the magnetic permeability of the streamline is constant.

PROOF : Using (3.7) in (1.12) we have

$$f(\eta)\frac{\partial h}{\partial\xi} = \frac{\partial}{\partial\xi}((\xi-\sigma))\left(\frac{\partial T}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\frac{\kappa}{\xi-\sigma}\frac{\partial T}{\partial\eta}\right)+\frac{r^4\mu_e\nu_H}{c^4(\xi-\sigma)}$$

(equation continued on p. 608)

$$\times \left[\frac{\partial}{\partial \eta} \left(\frac{2 f(\eta)}{\rho (\xi - \sigma)} \right) \right]^3 + \frac{f(\eta)}{2} \frac{\partial}{\partial \xi} \left(\frac{f^2(\eta) r^4}{\rho^2 c^4 (\xi - \sigma)^2} \right)$$

$$- \frac{\mu_e f(\eta)}{2\rho} \frac{\partial}{\partial \xi} \left(\frac{\alpha^2 f^2(\eta) r^4}{\rho^2 c^4 (\xi - \sigma)^2} \right) + \frac{2\nu_H r^2}{c (\xi - \sigma)} \frac{c^6}{r^6}$$

$$\times \left[\frac{\partial}{\partial \eta} \left(\frac{f(\eta) r^4}{\rho c^2 (\xi - \sigma)} \right) \right]^2 + \frac{2 f(\eta) r^2 c^2}{c^2 (\xi - \sigma)} \left[\frac{1}{r} - \frac{2 (\xi - \sigma)}{r^3} r_1 \right]$$

$$\times \left[- p + \frac{2\nu_H f(\eta) r^5 c^2}{\rho c^6 (\xi - \sigma)^2} \left(\frac{1}{r^2} - \frac{2 (\xi - \sigma)}{r^3} r_1 \right) \right]$$

$$+ \frac{2c^2}{r^2} (\xi - \sigma) \left[- p + \frac{2 r^2 \nu_H}{c^2} \frac{\partial}{\partial \xi} \left(\frac{f(\eta) r^2}{\rho c^2 (\xi - \sigma)} \right) \right]$$

$$\times \frac{\partial}{\partial \xi} \left(\frac{r^2 f(\eta)}{\rho c^2 (\xi - \rho)} \right). \qquad (3.8)$$

Simplifying the above equation we get

$$f(\eta) \frac{\partial h}{\partial \xi} = \frac{\partial \dot{T}}{\partial \xi} + (\xi - \sigma) \frac{\partial \kappa}{\partial \eta} \frac{\partial \dot{T}}{\partial \xi} + (\xi - \sigma) \kappa \frac{\partial^2 \dot{T}}{\partial \xi^2}$$

$$+ \frac{1}{\xi - \sigma} \frac{\partial \kappa}{\partial \eta} \frac{\partial \dot{T}}{\partial \eta} + \frac{\kappa \sigma'}{(\xi - \sigma)^2} \frac{\partial \dot{T}}{\partial \eta} + \frac{\kappa}{\xi - \sigma} \frac{\partial^2 \dot{T}}{\partial \eta^2}$$

$$+ \frac{r^4 \mu_e \nu_H}{c^4 (\xi - \sigma)} \left[\frac{f(\eta)}{\rho (\xi - \sigma)} \frac{\partial \alpha}{\partial \eta} + \frac{\alpha f'(\eta)}{\rho (\xi - \sigma)} + \frac{\alpha f(\eta) \sigma'(\eta)}{\rho (\xi - \sigma)^2} \right.$$

$$\left. - \frac{\alpha f(\eta)}{\rho^2 (\xi - \sigma)} \frac{\partial \rho}{\partial \eta} \right] + \frac{f^3(\eta)}{2c^4} \left[\frac{4 r^3 r_1}{\rho^2 (\xi - \sigma)^3} - \frac{2 r^4}{\rho^2 (\xi - \sigma)^3} \right.$$

$$\left. - \frac{2 r^4}{\rho^3 (\xi - \sigma)^2} \frac{\partial \rho}{\partial \xi} \right] - \frac{\mu_e f^3(\eta)}{2\rho c^4} \left[\frac{4 \alpha^2 r^3 r_1}{\rho (\xi - \sigma)^2} - \frac{2 r^4 \alpha^2}{\rho^2 (\xi - \sigma)^3} \right.$$

$$\left. - \frac{2 r^4 \alpha^2}{\rho^3 (\xi - \sigma)^2} \frac{\partial \rho}{\partial \xi} + \frac{2 \alpha r^4}{\rho^2 (\xi - \sigma)^2} \frac{\partial \alpha}{\partial \xi} \right] + \frac{2\nu_H}{(\xi - \sigma) r^4}$$

$$\times \left[\frac{f'(\eta) r^4}{\rho (\xi - \sigma)} + \frac{4 r^3 r_2 f(\eta)}{\rho (\xi - \sigma)} - \frac{f(\eta) r^4}{\rho^2 (\xi - \sigma)} \frac{\partial \rho}{\partial \eta} \right.$$

$$\left. + \frac{\sigma' f(\eta) r^4}{\rho (\xi - \sigma)^2} \right]^2 + \frac{2 f(\eta)}{\rho (\xi - \sigma)} \left[1 - \frac{2 (\xi - \sigma)}{(-\sigma)} r_1 \right]$$

$$\left[- p + \frac{2\nu_H f(\eta) r^1}{\rho c^4 (\xi - \sigma)^2} \left(1 - \frac{2 (\xi - \sigma)}{r} r_1 \right) \right]$$

$$+ \frac{2 (\xi - \sigma)}{r^2} f(\eta) \left[\frac{2 r r_1}{\rho (\xi - \sigma)} - \frac{r^2}{\rho^2 (\xi - \sigma)} \frac{\partial \rho}{\partial \xi} + \frac{r^2}{\rho (\xi - \rho)^2} \right]$$

(*equation continued on p. 609*)

$$\left[-p + \frac{2r^2}{c^4}\nu_H f(\eta)\left(\frac{2r\,r_1}{\rho\,(\xi-\sigma)} - \frac{r^2}{\rho^2\,(\xi-\sigma)}\frac{\partial\rho}{\partial\xi}\right.\right.$$

$$\left.\left. - \frac{r^2}{\rho\,(\xi-\sigma)^2}\right)\right. . \qquad \ldots(3.9).$$

Multiplying (3.9) by $(\xi - \rho)^5$, we get

$$(\xi-\sigma)^5 f(\eta)\frac{\partial h}{\partial\xi} = (\xi-\sigma)^5\,\kappa\,\frac{\partial \dot{T}}{\partial\xi} + (\xi-\sigma)^6\,\frac{\partial\kappa}{\partial\xi}\,\frac{\partial \dot{T}}{\partial\xi} + \kappa(\xi-\sigma)^6\,\frac{\partial^2 T}{\partial\xi^2}$$

$$+ (\xi-\sigma)^4\,\frac{\partial\kappa}{\partial\eta}\,\frac{\partial^2 T}{\partial\eta} + \kappa\,\sigma'\,(\xi-\sigma)^3\,\frac{\partial\dot{T}}{\partial\eta}$$

$$+ \kappa\,(\xi-\sigma)^4\,\frac{\partial^2 T}{\partial\eta^2} + \frac{r^4\,\mu_e\,\nu_H}{c^4}\times\left[(\xi-\sigma)\left(\frac{f(\eta)}{\rho}\,\frac{\partial\alpha}{\partial\eta}\right.\right.$$

$$\left.\left. + \frac{\alpha f'(\eta)}{\rho} - \frac{\alpha f(\eta)}{\rho^2}\,\frac{\partial\rho}{\partial\eta}\right) + \frac{\alpha f(\eta)\,\sigma'(\eta)}{\rho}\right]^3$$

$$+ \frac{f^3(\eta)}{c^4}\,(\xi-\sigma)^2\left[\frac{2r^3\,r_1}{\rho^2}(\xi-\sigma) - \frac{r^4}{\rho^2}\right.$$

$$\left. - \frac{r^4}{\rho^2}\,(\xi-\sigma)\,\frac{\partial\rho}{\partial\xi}\right] - \frac{\mu_e\,f^3(\eta)}{\rho c^4}\,(\xi-\sigma)^2$$

$$\times\left[\frac{r^4}{\rho^2}\left(2\alpha^2 r - \frac{r\alpha^2}{\rho}\,\frac{\partial\rho}{\partial\xi} + \alpha\,r\,\frac{\partial\alpha}{\partial\xi}\right)(\xi-\sigma)\right.$$

$$\left. - \frac{r^4\,\alpha^2}{\rho^2}\right] + \frac{2\nu_H}{r^4}\left[\frac{r^3}{\rho}\,(rf'(\eta) + 4r_2 f(\eta)\right.$$

$$\left. - \frac{rf(\eta)}{\rho}\,\frac{\partial\rho}{\partial\eta}\right)(\xi-\sigma) + \left.\frac{\sigma' f(\eta)\,r^4}{\rho}\right]^2$$

$$+ \frac{2f(\eta)}{\rho}\,(\xi-\sigma)^2\left[1 - \frac{2\,(\xi-\sigma)}{r}r_1\right]$$

$$\times\left[-p\,(\xi-\sigma)^2 + \frac{2\nu_H\,f(\eta)\,r^4}{c^4}\left(1 - \frac{2(\xi-\sigma)}{r}r_1\right)\right]$$

$$+ \frac{2f(\eta)\,(\xi-\sigma)^2}{r^2}\left[\frac{2rr_1}{\rho}\,(\xi-\sigma)\right.$$

$$\left. - \frac{r^2}{\rho^2}\,\frac{\partial\rho}{\partial\xi}\,(\xi-\sigma) - \frac{r^2}{\rho}\right]\left[-p\,(\xi-\sigma)^2 + \frac{2r^2\,\nu_H\,f(\eta)}{c^4}\right.$$

$$\left.\times\left(\frac{2r\,r_1}{\rho}\,(\xi-\sigma) - \frac{r^2}{\rho^2}(\xi-\sigma)\,\frac{\partial\rho}{\partial\xi} - \frac{r^2}{\rho^2}\right)\right]. \qquad \ldots(3.10)$$

Equation (3.10) holds identically on the curve $\xi = \sigma(\eta)$ and finally we must have

$$\frac{r^4 \mu_e \nu_H}{c^4} \frac{\alpha^2 f^2(\eta) \sigma'^2(\eta)}{\rho^2} + \frac{2\nu_H}{r^4} \frac{\sigma'^2 f^2(\eta) r^8}{\rho^2} = 0. \quad ...(3.11)$$

This implies that

$$\frac{r^4 \nu_H f^2(\eta) \sigma'^2(\eta)}{\rho^2}\left(\frac{\mu_e}{c^4} + 2\right) = 0.$$

Since $r \neq 2$, $\nu_H \neq 0$, $f(\eta) \neq 0$ and $\rho \neq 0$

we must have $\sigma' = 0$ or $\mu_e = -2c^4$.

Thus the image streamlines of S in the inverse surface $\bar{S}$ are concentric circles or the magnetic permeability of streamline is constant and it is equal to $-2c^4$.

4. The Study of Streamlines of MGD Flow of a Surface S in the Parallel Surface $\bar{S}$

A surface $\bar{S}$ which is at a constant distance along the normal from another surface S is said to be parallel to S. As the constant distance may be chosen arbitrarily the number of such parallel surfaces is infinite. If r is the current point on the surface S, n the unit normal to that surface and c the constant distance, the corresponding point $\bar{r}$ on the parallel surface $\bar{S}$ is given by

$$\bar{r} = r + cn. \quad ...(4.1)$$

Let the lines of curvature on S be taken as parametric curves. If α, β are the principal radii of curvature at the point r on S. The corresponding point A on the first sheet of the centre surface is

$$\bar{r} = r + \alpha n. \quad ...(4.2)$$

Now since A is the centre of curvature for η = constant, it follows that $\bar{r}$ and α are constant for one differentiation w.r.t ξ. Thus

$$r_1 + \alpha n_1 = 0. \quad ...(4.3)$$

Similarly

$$r_2 + \beta n_2 = 0. \quad ...(4.4)$$

Since the lines of curvature on S are taken as parametric curves, so that $F = 0$ and $M = 0$. Then by virtue of (4.3) and (4.4) we have from (4.2)

$$\bar{r}_1 = r_1 + cn_1 = -\frac{(c-\alpha)}{\alpha} r_1 \quad ...(4.5)$$

$$\bar{r}_2 = r_2 + cn_2 = -\frac{(c-\beta)}{\beta} r_2 . \quad ...(4.6)$$

The magnitudes of the first order for the parallel surface are given by

$$\left.\begin{aligned}\bar{E} &= \bar{r}_1 \cdot \bar{r}_1 = \left(\frac{c-\alpha}{\alpha}\right)^2 E,\ \bar{F} = \left(\frac{c-\alpha}{\alpha}\right)\left(\frac{c-\beta}{\beta}\right) r_1 \cdot r_2 \\ \bar{F} &= \left(\frac{c-\alpha}{\alpha}\right)\left(\frac{c-\beta}{\beta}\right) F = 0,\ \bar{G} = \frac{U}{r_2} \cdot \bar{r}_2 = \left(\frac{c-\beta}{\beta}\right)^2 G.\end{aligned}\right\} \qquad \ldots(4.7)$$

Assume streamlines on the surface S to be straight lines but not parallel. By knowing the relations between the fundamental magnitudes of S and $\bar{S}$ we study the flow of MGD of surface S on the parallel surface $\bar{S}$.

Using (1.14) in (4. 7) we have

$$\bar{E} = \left(\frac{c-\alpha}{\alpha}\right)^2, \bar{F} = 0, \bar{G} = \left(\frac{c-\beta}{\beta}\right)^2 (\xi - \sigma(\eta))^2 \qquad \ldots(4.8)$$

where α, β are principal radii and $k_a\ \alpha = 1$, $k_b\ \beta = 1$ and* k_a, k_b are principal curvatures.

Let $\bar{E} = g_1^2\,(\xi, \eta)\,(ck_a - 1)^2$, $F = 0$,

$\bar{G} = (ck_b - 1)^2\,(\xi - \sigma\,(\eta))^2$.

Then g_1 and g_2 for $\bar{S}$ are given by

$$g_1\,(\xi, n) = (ck_a - 1),\ g_2\,(\xi, \eta) = (ck_b - 1)\,(\xi - \sigma\,(\eta)). \qquad \ldots(4.9)$$

We know that the flow in S is planar. The Gauss equation corresponding to the fundamental magnitudes of $\bar{S}$, is given by (3.6).

Using (4.9) in (3.6) we have

$$\begin{aligned}\bar{K} = \frac{-1}{g_1 g_2}\Bigg[&\frac{\partial}{\partial\xi}\left(\frac{1}{(ck_a-1)}\ \frac{\partial\,(ck-1)\,(\xi-\sigma(\eta))}{\partial\tau}\right) \\ &+ \frac{\partial}{\partial\eta}\left(\frac{1}{(ck_b-1)\,(\xi-\sigma(\eta))} \times \frac{\partial\,(ck_a-1)}{\partial\eta}\right)\Bigg] \\ = \frac{-1}{g_1 g_2}\Bigg[&\frac{\partial}{\partial\xi}\left(\frac{1}{(ck_a-1)}\left(c\,\frac{\partial k_b}{\partial\xi}\left(\xi-\sigma(\eta)\right) + (ck_b-1)\right)\right) \\ &+ \frac{\partial}{\partial\eta}\left(\frac{1}{(ck_b-1)\,(\xi-\sigma(\eta))}\ c\,\frac{\partial k_a}{\partial n}\right)\Bigg].\end{aligned}$$

Use (1.14) in (2.14) we get $\quad \bar{K} = 0$.

Hence the flow in the parallel surface $\bar{S}$ of S is planar.

Theorem 4.1—If the streamlines in MGD in the surface S are straight lines but not parallel and envelope a curve C and the tangent lines to C i.e. η = constant and the involute of the curve C i.e. ξ = constant are taken as orthogonal curvilinear co-

*Note : From now onwards treat $k_a = k_1$, $k_b = k_2$.

ordinates. Then the image streamlines of S in the parallel surface $\bar{S}$ of S are either concentric circles or the magnetic permeability of the streamline is constant.

PROOF : Using (4.9) in (1.12) we have

$$f(\eta)\frac{\partial h}{\partial \xi}=\frac{\partial}{\partial \xi}\left(\frac{\kappa}{ck_1-1}\frac{\partial T}{\partial \xi}(ck_2-1)\,\xi-\sigma(\eta)\right)$$

$$+\frac{\partial}{\partial \eta}\left(\frac{\kappa}{(ck_2-1)(\xi-\sigma(\eta))}\frac{\partial T}{\partial \eta}(ck_1-1)\right)$$

$$+\frac{\mu_e\,\nu_H}{(ck_a-1)(ck_b-1)(\xi-\sigma(\eta))}\left[\frac{\partial}{\partial \eta}\left(\frac{(ck_a-1)\,\alpha f(\eta)}{\rho(ck_b-1)(\xi-\sigma)}\right)\right]^2$$

$$+\frac{f(\eta)}{2}\frac{\partial}{\partial \xi}\left[\frac{f^2(\eta)}{\rho^2(\xi-\sigma)^2(ck_b-1)^2}\right]-\frac{\mu_e f(\eta)}{2\rho}\frac{\partial}{\partial \xi}$$

$$\times\left(\frac{\alpha^2 f^2(\eta)}{\rho^2(ck_b-1)^2(\xi-\sigma)^2}\right)+\frac{2\nu_H(ck_a-1)^3}{(ck_b-1)(\xi-\sigma)}$$

$$\times\left[\frac{\partial}{\partial \eta}\left(\frac{f(\eta)}{(ck_a-1)(ck_b-1)(\xi-\sigma)}\right)\right]^2+\frac{2f(\eta)}{\rho(ck_b-1)(\xi-\sigma)}$$

$$\times\left[-p+\frac{2\nu_H f(\tau)}{\rho(ck_1-1)(ck_2-1)^2(\xi-\sigma)^2}\frac{\partial}{\partial \xi}(ck_b-1)(\xi-\sigma))\right]$$

$$\times\frac{\partial}{\partial \xi}((ck_b-1)(\xi-\sigma))+2(ck_b-11)(\xi-\sigma)$$

$$\times\left[-p+\frac{2\nu_H}{ck_a-1}\frac{\partial}{\partial \xi}\left(\frac{f(\eta)}{(ck_b-1)(\xi-\sigma)}\right)\right]$$

$$\times\frac{\partial}{\partial \xi}\left(\frac{f(\eta)}{\rho(ck_b-1)(\xi-\sigma)}\right). \qquad \ldots(4.10)$$

Using (4.9) and (2.14) in (4.10) we get

$$f(\eta)\frac{\partial h}{\partial \xi}=\frac{(ck_b-1)(\xi-\sigma(\eta))}{(ck_a-1)}\frac{\partial \kappa}{\partial \xi}\frac{\partial \dot{T}}{\partial \xi}+\frac{\kappa(ck_b-1)(\xi-\sigma)}{(ck_a-1)}\frac{\partial^2 \dot{T}}{\partial \xi^2}$$

$$+\frac{\kappa(ck_b-1)}{(ck_a-1)}\frac{\partial \dot{T}}{\partial \xi}+\frac{c\,\kappa(\xi-\sigma)}{(ck_a-1)}\frac{\partial \dot{T}}{\partial \xi}\frac{k_a-k_b}{(\xi-\sigma)}$$

$$-\frac{\kappa c(ck_b-1)(\xi-\sigma)}{(ck_1-1)^2}\frac{\partial k_a}{\partial \xi}\frac{\partial \dot{T}}{\partial \xi}+\frac{\kappa(ck_a-1)}{(ck_b-1)(\xi-\sigma)}\frac{\partial^2 \dot{T}}{\partial \eta^2}$$

$$+\frac{(ck_a-1)}{(ck_b-1)(\xi-\sigma)}\frac{\partial \kappa}{\partial \eta}\frac{\partial \dot{T}}{\partial \eta}-\frac{c\kappa(ck_a-1)}{(ck_b-1)^2(\xi-\sigma)}\frac{\partial \dot{T}}{\partial \eta}\frac{\partial k_2}{\partial \eta}$$

(equation continued on p. 613)

$$+ \frac{\kappa\,(ck_a - 1)\,\sigma'(\eta)}{(ck_b - 1)(\xi - \sigma)^2}\,\frac{\partial T}{\partial \eta} + \frac{\mu_e\,\nu_H}{(ck_a - 1)(ck_b - 1)(\xi - \sigma)}$$

$$\times \left[\frac{\partial}{\partial \eta}\left(\frac{f(\eta)}{\rho}\right) \frac{ck_a - 1}{(ck_b - 1)(\xi - \sigma)} - \frac{\kappa\,(ck_a - 1)\,\sigma'(\eta)}{(ck_b - 1)(\xi - \sigma)^2}\,\frac{\partial T}{\partial \eta} \right]$$

$$+ \frac{\mu_e\,\nu_H}{(ck_a - 1)(ck_b - 1)(\xi - \sigma)} \left[\frac{\partial}{\partial \eta}\left(\frac{f(\eta)}{\rho}\right) \frac{ck_a - 1}{(ck_b - 1)(\xi - \sigma)} \right.$$

$$+ \frac{\alpha f(\eta)\,c}{\rho\,(ck_b - 1)(\xi - \sigma)}\,\frac{\partial k_1}{\partial \eta} - \frac{c\,(ck_a - 1)\,\alpha f(\eta)}{\rho\,(ck_b - 1)^2(\xi - \sigma)}\,\frac{\partial k_b}{\partial \eta}$$

$$\left. + \frac{\sigma'(\eta)\,(ck_a - 1)\,\alpha f(\eta)}{\rho\,(ck_a - 1)(\xi - \sigma)^2} \right]^2 + \frac{2\nu_H\,(ck_a - 1)^3}{(ck_b - 1)(\xi - \sigma)}$$

$$\times \left[\frac{\partial}{\partial \eta}\left(\frac{f(\eta)}{\rho}\right) \frac{1}{(ck_a - 1)(ck_b - 1)(\xi - \sigma)} \right.$$

$$+ \frac{f(\eta)\,\sigma'(\eta)}{(ck_a - 1)(ck_b - 1)(\xi - \sigma)^2} - \frac{cf(\eta)}{\rho(ck_a - 1)^2(ck_b - 1)(\xi - \sigma)}\,\frac{\partial k_1}{\partial \eta}$$

$$\left. - \frac{cf(\eta)}{\rho\,(ck_a - 1)(ck_2 - 1)^2(\xi - \sigma)}\,\frac{\partial k_2}{\partial \eta} \right]^2 + \frac{2f(\eta)}{\rho(ck_b - 1)(\xi - \sigma)}$$

$$\times \left[-p + \frac{2\nu_H\,f(n)}{(ck_a - 1)(ck_b - 1)^2(\xi - \sigma)^2} \left\{ c\,(\xi - \sigma)\,\frac{\partial k_b}{\partial \xi} \right. \right.$$

$$\left. \left. + (ck_b - 1) \right\} \right] \left[c\,(\xi - \sigma)\,\frac{\partial k_b}{\partial \xi} + (ck_b - 1) \right]$$

$$+ 2(ck_b - 1)(\xi - \sigma) \left[\frac{\partial}{\partial \xi}\left(\frac{f(\eta)}{\rho}\right) \frac{1}{(ck_b - 1)(\xi - \sigma)} \right.$$

$$- \frac{cf(\eta)}{\rho(ck_b - 1)^2(\xi - \sigma)}\,\frac{\partial k_2}{\partial \xi} - \frac{c(ck_a - 1)\,\alpha f(\eta)}{\rho(ck_b - 1)^2(\xi - \sigma)}\,\frac{\partial k_b}{\partial \eta}$$

$$\left. + \frac{\sigma'(\eta)\,(ck_a - 1)\,\alpha f(\eta)}{\rho(ck_b - 1)(\xi - \sigma)^2} \right]^2 + \frac{2\nu_H\,(ck_a - 1)^3}{(ck_b - 1)(\xi - \sigma)}$$

$$\times \left[\frac{\partial}{\partial \eta}\left(\frac{f(\eta)}{\rho}\right) \frac{1}{(ck_a - 1)(ck_b - 1)(\xi - \sigma)} \right.$$

$$+ \frac{f(\eta)\,\sigma'(\eta)}{\rho(ck_a - 1)(ck_b - 1)(\xi - \sigma)^2} - \frac{cf(\eta)}{\rho(ck_1 - 1)(ck_b - 1)^2(\xi - \sigma)}$$

$$\left. \times \frac{\partial k_2}{\partial \eta} \right]^2 + \frac{2f(\eta)}{\rho\,(ck_b - 1)(\xi - \sigma)} \left[-p + \frac{2\nu_H\,f(\eta)}{\rho(ck_b - 1)^2(\xi - \sigma)^2} \right.$$

$$\times (ck_a - 1) + 2(ck_b - 1)(\xi - \sigma) \left[\frac{\partial}{\partial \xi}\left(\frac{f(\eta)}{\rho}\right) \right.$$

(*equation continued on p. 614*)

$$\times \frac{1}{(ck_2 - 1)(\xi - \sigma)} - \frac{cf(\eta(k_a - k_b)}{\rho(ck_2 - 1)^2 (\xi - \sigma)^2}$$

$$- \frac{f(\eta)}{\rho(ck_2 - 1)(\xi - \sigma)^2}\Bigg]\Bigg[-p + \frac{2\nu_H}{(ck_a - 1)}$$

$$\times \left\{\frac{-f(\eta)}{\rho(ck_b - 1)(\xi - \sigma)^2} - \frac{cf(\eta)(k_a - k_b)}{\rho(ck_b - 1)^2 (\xi - \sigma)^2}\right.$$

$$\left. + \frac{\partial}{\partial \xi}\left(\frac{f(\eta)}{\rho}\right)\frac{1}{(ck_b - 1)(\xi - \sigma)}\right\}\Bigg]. \qquad \ldots(4.11)$$

Multiplying (4.11) by $(\xi - \sigma(\eta))^5$, then we have

$$(\xi - \sigma(\eta))^5 f(\eta) \frac{\partial h}{\partial \xi} = \frac{(ck_b - 1)}{(ck_a - 1)} (\xi - \sigma(\eta))^6 \frac{\partial \kappa}{\partial \xi} \frac{\partial \dot{T}}{\partial \xi}$$

$$+ \frac{\kappa (ck_b - 1)(\xi - \sigma)^6}{(ck_a - 1)} \frac{\partial^2 \dot{T}}{\partial \xi^2} + (\xi - \sigma)^5 \kappa \frac{\partial T}{\partial \xi}$$

$$- \frac{\kappa c(ck_b - 1)}{(ck_a - 1)^2} (\xi - \sigma)^6 \frac{\partial k_a}{\partial \xi} \frac{\partial \dot{T}}{\partial \xi} + \frac{\kappa (ck_a - 1)}{(ck_b - 1)}$$

$$\times (\xi - \sigma)^4 \frac{\partial^2 T}{\partial \eta^2} + \frac{(ck_a - 1)}{(ck_b - 1)} (\xi - \sigma)^4 \frac{\partial \kappa}{\partial \eta} \frac{\partial T}{\partial \eta}$$

$$- \frac{c\kappa (ck_a - 1)(\xi - \sigma)^4}{(ck_b - 1)^2} \frac{\partial \dot{T}}{\partial \eta} \frac{\partial k_b}{\partial \eta}$$

$$+ \frac{\kappa (ck_a - 1)\sigma'(\eta)(\xi - \sigma)^3}{(ck_b - 1)} \frac{\partial T}{\partial \eta}$$

$$+ \frac{\mu_e \nu_H}{(ck_a - 1)(ck_b - 1)}\left[\frac{\partial}{\partial \eta}\left(\frac{f(\eta)}{\rho}\right)\frac{ck_a - 1}{ck_b - 1}(\xi - \sigma)\right.$$

$$- \frac{c(ck_a - 1)\alpha f(\eta)}{\rho(ck_b - 1)^2} (\xi - \sigma) \frac{\partial k_2}{\partial \eta}$$

$$\left. + \frac{\sigma'(\eta)(ck_a - 1)\alpha f(\eta)}{\rho(ck_b - 1)}\right]^2 + \frac{2\nu_H (ck_a - 1)^3}{(ck_b - 1)}$$

$$\times \left[\frac{\partial}{\partial \eta}\left(\frac{f(\eta)}{\rho}\right)\frac{(\xi - \sigma)}{(ck_a - 1)(ck_b - 1)}\right.$$

$$\left. + \frac{f(\eta)\sigma'(\eta)}{\rho(ck_a - 1)(ck_b - 1)} - \frac{cf(\eta)(-)}{\rho(ck_a - 1)(ck_b - 1)^2}\frac{\partial k_b}{\partial \eta}\right]^2$$

$$+ \frac{2f(\eta)(\xi - \sigma)^2}{\rho(ck_b - 1)}\left[-p(\xi - \sigma(\eta))^2 + \frac{2\nu_H f(\eta)}{(ck_b - 1)^2}\right]$$

(*equation continued on p. 615*)

$$\times (ck_a-1) + 2(ck-1)(\xi-\sigma)^2 \left[\frac{\partial}{\partial\xi}\left(\frac{f(\eta)}{\rho}\right)\frac{(\xi-\sigma)}{(ck_b-1)} - \frac{cf(\eta)(ck_a-1)}{\rho(ck_b-1)^2}\right]\left[-p(\xi-\sigma)^2 + \frac{2\nu_H}{(ck_a-1)} \times \left\{\frac{(\xi-\sigma)}{ck_b-1}\frac{\partial}{\partial\xi}\left(\frac{f(\eta)}{\rho}\right) - \frac{f(\eta)(ck_a-1)}{\rho(ck_b-1)^2}\right\}\right]. \quad ...(4.12)$$

The above equation (4.12) holds identically on the curve $\xi = \sigma(\eta)$. Finally we must have

$$\frac{\mu_e \nu_H}{(ck_a-1)(c_b k_b-1)}\left(\frac{\sigma'(\eta)(ck_a-1)\alpha f(\eta)}{\rho(ck_b-1)}\right)^2 + \frac{2\nu_H(ck_a-1)^3}{(ck_b-1)} \times \left(\frac{f(\eta)\sigma'(\eta)}{\rho(ck_a-1)(ck_b-1)}\right)^2 = 0.$$

The above equation implies

$$\frac{\mu_e \nu_H}{(ck_a-1)(ck_b-1)}\frac{(\sigma'(\eta))^2(ck_a-1)^2\alpha^2 f^2(\eta)}{\rho^2(ck_b-1)^2} + \frac{2\nu_H(ck-1)^3}{(ck_b-1)} \times \frac{f^2(\eta)(\sigma'(\eta))^2}{\rho^2(ck_a-1)^2(ck_b-1)} = 0$$

implies

$$\frac{(ck_a-1)}{(ck_b-1)^3\rho^2} f^2(\eta)\sigma'^2(\eta)\nu_H(\mu_e+2) = 0.$$

Since $f(\eta) \neq 0$, $\nu_H \neq 0$, $\rho \neq 0$ and $\dfrac{ck_a-1}{(ck_b-1)^3} \neq 0$

we must have

$$\sigma'^2(\eta)(\mu_e+2) = 0. \quad ...(4.13)$$

This equation (4.13) implies

$$\sigma'(\eta) = 0 \text{ or } \mu_t + 2 = 0.$$

Hence the image streamlines of S in $\bar{S}$ are concentric circles or the magnetic permeability of streamline is constant and it is equal to -2.

REFERENCES

1. O. P. Chandna, and M. R. Garg, *Quart. Appl. Math.* 34 (1976) 287-300.
2. V. I. Nath, and O. P. Chandna, *Quart. Appl. Math.* 31 (1973), 351-362.
3. V. I. Nath, and O. P. Chandna, *Quart. Appl. Math.* 32 (1974) 337-341.
4. H. C. Sinha, *Three Dimensional Differential Geometry*, S. Chand & Co. (Pvt.) Ltd., 1972.

5. C. E. Weatherburn, *Differential Geometry of Three Dimensions*. ELBS and Cambridge Univ. Press, 1947.
6. S. I. Par. *Magnetogasdynamics and Plasma Dynamics*. Wein. Springer-Verlag, 1962.
7. I. S. Sokolnikoff, *Tensor Analysis*, John Wiley and Sons, Inc., and Toppan Company Ltd , 1966.
8. R. J. Cole, *Vector methods*. Van Nostrand Reinhold Company, 1974.

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